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Habilitation Thesis

Contributions to Brown representability problem

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Introduction

The formulation of one or another incarnation of Brown representability theorem may be technical, but the idea behind all of them is simple and elegant: We want to be able to construct right or left adjoints for triangulated functors preserving coproducts, respectively products; because the categories we work with do not satisfy the hypothesis of Freyd's Adjoint Functor Theorem we need a replacement for it in the new setting. This is Brown representability.

The problem of the existence of the adjoint functors and the one of representability of a given functor are strongly related (to fix the settings, suppose that we work with preadditive categories): In one direction, a functor $F : \mathcal{C} \to \mathcal{D}$ has a left adjoint if and only if the functor $\mathcal{D}(D, F(-)) : \mathcal{C} \to \mathcal{A}b$ is representable for all $D \in \mathcal{D}$ (here and overall $\mathcal{A}b$ denotes the category of abelian groups). Conversely a functor $F : \mathcal{C} \to \mathcal{A}b$ has a left adjoint if and only if it is representable (actually it is represented by the left adjoint evaluated at $\mathbb{Z}$, see [22, p. 81-82]).

The name Brown representability comes from Edgar H. Brown which proved in [14] that the homotopy category of spectra satisfies this property. Afterwards a general version at the level of triangulated categories is due to Amnon Neeman in his influential book on this subject [60]. This version, which is recalled in Preliminaries, is the one we work with. In the same book [60], it is shown that so called well-generated triangulated categories satisfy Brown representability. The class of well-generated triangulated categories is quite large, being closed under localization. Therefore since the derived category of modules over an arbitrary ring is well–generated (even compactly generated), the same should be true for the derived category of a Grothendieck abelian category. Despite the fact that compactly generated categories satisfies the dual of Brown representability too, this is not more automatically true for a localization. Neeman considers that the main problem which remained open in his book is to establish which categories satisfy the dual of the Brown representability. A strong impetus for the study of Brown representability was provided by the fact that Neeman applied it in algebraic geometry, for giving in [59] a conceptual proof for Grothendieck duality based on the existence of adjoint functors.

The present work records some progresses in the study of Brown representability which were obtained along almost ten years. The results here were first published in author's papers [9], [49], [50], [51], [52], [53], [54] and [56]. The papers [9] and [56] are joint works with Simion Breaz, respectively with Jan Šťovíček. It is a pleasure to thank them both for kindly agreeing with the use of these papers in this thesis. It should also be noted that putting together results published in various places and times these results can not be simply pasted one after other. Besides notational compatibility, the stuff was arranged in the logical order which does not always coincide with the chronological order. In the beginning of each chapter it is indicated the papers where were published results contained in that chapter. When some important changes were operated, this fact is also briefly explained.

In the first Chapter are defined main notions used overall in this work, and are collected some preliminary results. More details can be found as follows: for modules over rings with several objects in [39] and [27], for triangulated categories in [60] and [44], for general and abelian categories in [67], for homotopy and derived category in [35], [36] or [79].

Chapter 2, titled *Abelianization* deals with the category $\mathrm{mod}(\mathcal{T})$ of finitely presented contravariant functors from a triangulated category $\mathcal{T}$ to $\mathcal{A}b$. It is given a reformulation of Brown representability at the level of $\mathrm{mod}(\mathcal{T})$. This reformulation is used in order to give a new, more conceptual proof for a theorem due to Heller: A homological product preserving functor $F : \mathcal{T} \to \mathcal{A}b$ is representable if and only if it has a solution object.

Each of the next two chapters contains a new method for proving Brown representability. In the Chapter 3 is introduced a key property of our approach, namely *Deconstructibility in triangulated categories*. In analogy with the case of abelian categories, a triangulated category $\mathcal{T}$ is deconstructible if there is a set of objects $\mathcal{S}$ with the property that any object $X \in \mathcal{T}$ can be obtained, up to isomorphism, as the homotopy colimit of a tower consisting of maps whose mapping cones are direct summands of arbitrary direct sums of objects in $\mathcal{S}$. Further it is shown that deconstructible triangulated categories satisfies Brown representability. A main advantage of this approach is that all results can be immediately dualized, therefore we obtain a criterion for the dual of Brown representability too. As we expect, well–generated triangulated categories are deconstructible.

In Chapter 4 are introduced *Quasi-locally presentable categories*. Roughly speaking they are big unions of locally presentable categories in the sense of [1] and their definition is an axiomatization of some properties of the abelianization of a well–generated triangulated category. The intention is to change a little the perspective that the abelianization is to big, in the sense that it is not well (co)powered (see [60, Appendix C]), hence not manageable (see also [60, Remark 5.3.10] and the Introduction of Krause's work [44]). More exactly, one important result about the existence of adjoints depends on the categories being well powered, namely the special Freyd's adjoint functor theorem: if $\mathcal{C}$ is a complete, well powered category having a cogenerator, then every functor $F : \mathcal{C} \to \mathcal{D}$ has a left adjoint if and only if it preserves limits, see [22, p. 89]. We argue that even if the abelianization of a well generated triangulated category is not well (co)powered, it has enough structure allowing us to apply the general Freyd's adjoint functor theorem: if $\mathcal{C}$ is a complete category, then every functor $F : \mathcal{C} \to \mathcal{D}$ has a left adjoint if and only if it preserves limits and satisfies the solution set condition (that is for every $D \in \mathcal{D}$ there is a set maps

$f_i : D \to F(C_i)$, $i \in I$ in $\mathcal{D}$, where $C_i \in \mathcal{C}$, such that every map $f : D \to F(C)$, with $C \in \mathcal{C}$, factors through as $f = F(k)f_i$ for some $k : C_i \to C$ in $\mathcal{C}$; see [1, 0.7]).

Chapter 5 deals with *Homotopy category of complexes.* It is shown that if $\mathcal{A}$ is a sufficiently nice additive category, then $\mathbf{K}(\mathcal{A})$ satisfies Brown representability if and only if there is a fixed object $X \in \mathcal{A}$ such that every object in $\mathcal{A}$ is direct summand of arbitrary direct sums of X. Again the result is dualizable. In particular, if R is a ring, then $\mathbf{K}(\mathrm{Mod}(R))$ satisfies Brown repesentability exactly if R is pure semisimple. Some examples of triangulated coproduct preserving functors without right adjoint are also provided.

In the Chapter 6, titled *Brown representability for the dual* it is applied the deconstructibility criterion for showing that the opposite of the following categories satisfies Brown representability: the derived category over a wide class of abelian categories $\mathcal{A}$, including the category of quasi-coherent sheaves over a finite dimensional projective space, the homotopy category of projective modules over a ring with several objects R and the homotopy category of projective representations by R-modules of a quiver. Therefore in some particular cases we found a solution of the Neeman unsolved problem about Brown representability for the dual. It is also worthy to note that the homotopy category of projective modules $\mathbf{K}(\mathrm{Proj}(R))$ plays a central role in [62], where it is also proved that it is generally $\aleph_1$-compactly generated but not ($\aleph_0$-)compactly generated. Until [52] where the results appear for the first time, there was known no other concrete example of a triangulated category which is not compactly generated and whose dual satisfies Brown representability.

The Appendix contains some problems which will be the subject of a further research. To make the reading easier, an Index is included at the end of this work.

All categories we work with are preadditive and all functors are additive. Generalities about categories which are not defined in this work can be found in monographs as [48] or [67]. I want also to mention that in the body of the work some definitions are numbered other are not. The difference between them is the following: Definitions of those notions which are used in more than one chapter are numbered. For the rest of the notions, the definition is incorporated in the text in order to increase its fluency.

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Chapter 1

Preliminaries

This is an introductory chapter, where we collect some well–known results which we need further. Proofs are only included when we consider they are relevant for further developments.

1.1 Modules over preadditive categories

Consider a *preadditive category* $\mathcal{T}$, that is a category whose hom–sets are endowed with a structure of abelian groups (with respect to an operation denoted with $+$) such that the composition is bilinear. We write $\mathcal{T}(X, X')$ for the abelian group of morphisms between X and X' in $\mathcal{T}$. Note that a preadditive category with exactly one object is nothing else than an associative ring with one.

Definition 1.1.1. By a *right module over* $\mathcal{T}$ (for shortly $\mathcal{T}$*-module*) we understand a contravariant functor $\mathcal{T} \to \mathcal{A}b$. In this work modules will always be at right, so for dealing with a *left* $\mathcal{T}$*-module* we have to consider a right $\mathcal{T}^o$-module, that is a functor $\mathcal{T} \to \mathcal{A}b$. The class of all $\mathcal{T}$-modules forms an abelian AB5 category, the morphisms being natural transformations, category which is denoted here by $\mathrm{Mod}(\mathcal{T})$.

Note that the limits and colimits in $\mathrm{Mod}(\mathcal{T})$ are computed point-wise. Usually, this category has no small Hom-sets (a category has small hom-sets if it lives in the universe we work in, therefore usually $\mathrm{Mod}(\mathcal{T})$ lives in a higher universe that $\mathcal{T}$), unless $\mathcal{T}$ is essentially small (i.e. it has a small skeleton). Skeletally small preadditive categories are also called rings with several objects. Note that categories we work with have small hom-sets, even if this is not clear from the beginning. Therefore we will not work with $\mathrm{Mod}(\mathcal{T})$ for those categories $\mathcal{T}$ which are not skeletally small. Instead of it we consider the full subcategory $\mathrm{mod}(\mathcal{T})$ of $\mathrm{Mod}(\mathcal{T})$, consisting of those $\mathcal{T}$-modules M which are *finitely presentable*, that is, there is an exact sequence

$$\mathcal{T}(-, X) \to \mathcal{T}(-, Y) \to F \to 0,$$

with $X, Y \in \mathcal{T}$. This last category has small Hom-sets, provided that $\mathcal{T}$ does, as we can see by Yoneda lemma. The Yoneda embedding

$$\mathcal{T} \to \mathrm{Mod}(\mathcal{T}),\ X \mapsto \mathcal{T}(-, X)$$

restricts to a well defined fully faithful functor

$$H = H_{\mathcal{T}} : \mathcal{T} \to \mathrm{mod}(\mathcal{T}),\ H(X) = \mathcal{T}(-, X),$$

called also Yoneda functor, or Yoneda embedding (we will omit the index $\mathcal{T}$ if no confusion is possible). We denote $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ the class of all natural transformations between two $\mathcal{T}$-modules.

Define also $\mathrm{mop}(\mathcal{T}) = (\mathrm{mod}(\mathcal{T})^o)^o$, and denote

$$H' = H'_{\mathcal{T}} : \mathcal{T} \to \mathrm{mop}(\mathcal{T}),\ H'(x) = \mathcal{T}(x, -).$$

For the sake of clarity we will denote by $\mathcal{T}(X, -)$ the respective (projective) object of $(\mathrm{mod}(\mathcal{T}))^o$ and by $H'(X)$ the same (injective) object viewed in $\mathrm{mop}(\mathcal{T})$. It is well known, that $\mathrm{mod}(\mathcal{T})$ (respectively $\mathrm{mop}(\mathcal{T})$) is an additive finitely cocomplete (complete) category with enough projectives (injectives), and any functor $F : \mathcal{T} \to \mathcal{A}$, into an additive finitely cocomplete (complete) category, extends uniquely, up to a natural isomorphism, to a cokernel (kernel) preserving functor

$$F_* : \mathrm{mod}(\mathcal{T}) \to \mathcal{A} \quad (F^* : \mathrm{mop}(\mathcal{T}) \to \mathcal{A}),$$

such that $F \cong F_* \circ H$ $(F \cong F^* \circ H')$. Often the category $\mathcal{A}$ is chosen to be abelian. Obviously, H commutes with products, and H' commutes with coproducts which exists in $\mathcal{T}$. If, in addition, $\mathcal{T}$ has coproducts (respectively, products), then $\mathrm{mod}(\mathcal{T})$ $(\mathrm{mop}(\mathcal{T}))$ has also coproducts (products), and the embedding H (H') commutes with coproducts (products). If this is the case, the a functor $F : \mathcal{T} \to \mathcal{A}$ preserves coproducts (products) if and only if the induced functor F_* (F^*) has the same property. Together with the observation that F_* (F^*) is always cokernel (kernel) preserving, this is further equivalent to the fact that F_* (F^*) preserves colimits (limits).

Definition 1.1.2. A morphism $X \to Y$ in has a *weak (co)kernel* if there is a morphism $X' \to Y$ $(Y \to Y')$ such that the sequence of $\mathcal{A}b$–valued functors

$$\mathcal{T}(-, X') \to \mathcal{T}(-, X) \to \mathcal{T}(-, Y),$$

$$(\text{ respectively } \mathcal{T}(Y', -) \to \mathcal{T}(Y, -) \to \mathcal{T}(-, X)\ )$$

is exact.

Recall that $\mathrm{mod}(\mathcal{T})$ $(\mathrm{mop}(\mathcal{T}))$ is abelian, provided that $\mathcal{T}$ has weak-kernels (weak-cokernels).

Definition 1.1.3. Let $F : \mathcal{T} \to \mathcal{A}b$ be a functor. The *comma category* of objects over F, denoted by $\mathcal{T}/F$, has as objects pairs of the form (X, x) where $X \in \mathcal{T}$ and $x \in F(X)$, and a map between (X, x) and (Y, y) in $\mathcal{T}/F$ is a map $f : X \to Y$ in $\mathcal{T}$ such that $F(f)(x) = y$.

Recall that the solution set condition for functors with values in the category of abelian groups $F : \mathcal{T} \to \mathcal{A}b$ can be stated as follows: There is a set $\mathcal{S}$ of objects in $\mathcal{T}$, such that for any $K \in \mathcal{T}$ and any $y \in F(K)$ there are $S \in \mathcal{S}$, $x \in F(S)$ and $f : S \to K$ satisfying $F(f)(x) = y$ (see [46, Chapter V, §6, Theorem 3]). We can reformulate this by saying that the category

$$\mathcal{S}/F = \{(S, x) \mid S \in \mathcal{S}, x \in F(S)\}$$

is *weakly initial* in $\mathcal{T}/F$, that is for every $(K, y) \in \mathcal{T}/F$ there exists a map $(S, x) \to (K, y)$ for some $(S, x) \in \mathcal{S}/F$. Via Yoneda lemma, every object $(S, x) \in \mathcal{S}/F$ corresponds to a natural transformation $\mathcal{T}(S, -) \to F$. In these terms, the existence of a solution set is further equivalent to the fact that there are objects $S_i \in \mathcal{T}$ indexed over a set I and a functorial epimorphism

$$\bigoplus_{i\in I} \mathcal{T}(S_i, -) \to F \to 0.$$

Definition 1.1.4. We say that $F : \mathcal{T} \to \mathcal{A}b$ has a *solution object* provided that there is an object $S \in \mathcal{T}$ and a functorial epimorphism

$$\mathcal{T}(S, -) \to F \to 0,$$

or equivalently, the category $\mathcal{T}/F$ has a weakly initial object.

Note that if there are arbitrary products in $\mathcal{T}$, and the functor F preserves them, then the existence of a solution set is clearly equivalent to that of a solution object. Obviously if $F \cong \mathcal{T}(S, -)$ is representable, then F has a solution object.

Let $\mathcal{A}$ be a additive category and $\mathcal{C} \subseteq \mathcal{A}$ be a subcategory. Let $F : \mathcal{A} \to \mathcal{A}b$ be a contravariant functor. The we can consider the comma category of objects over $F|_\mathcal{C}$, where $F|_\mathcal{C}$ denotes the restriction of F at $\mathcal{C}$, that is

$$\mathcal{C}/F = \{(X, x) \mid X \in \mathcal{C}, x \in F(C)\},$$

with the morphisms

$$\mathcal{C}/F((X_1, x_1), (X_2, x_2)) = \{\alpha \in \mathcal{C}(X_1, X_2) \mid F(\alpha)(x_2) = x_1\}.$$

In particular, for any object $A \in \mathcal{A}$, let denote

$$\mathcal{C}/A = \mathcal{C}/\mathcal{A}(-, A) = \{(C, \xi) \mid C \in \mathcal{C}, \xi : C \to A\},$$

$$\mathcal{C}/A((C_1, \xi_1), (C_2, \xi_2)) = \{\alpha \in \mathcal{C}(C_1, C_2) \mid \xi_2\alpha = \xi_1\}.$$

1.2 Triangulated categories

An *additive category* is a preadditive one, with zero object and finite biproducts.

Definition 1.2.1. A *triangulated category* is an additive category $\mathcal{T}$ endowed with an autoequivalence $\Sigma : \mathcal{T} \to \mathcal{T}$ called *shift functor* and a class of sequences of the form

$$X \to Y \to Z \to \Sigma X,$$

with the property that the composition of any two succesive morphisms vanishes which are called *triangles*. The class of triangles must be closed under isomorphisms and it is the subject of the following axioms:

T0. For every $X \in \mathcal{T}$ the sequence $X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow \Sigma X$ is a triangle.

T1. Every map $X \to Y$ may be completed to a triangle $X \to Y \to Z \to \Sigma X$.

T2. If $\mathcal{X} \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is a triangle then so are

$$Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma X \text{ and } \Sigma^{-1} Z \xrightarrow{-\Sigma^{-1} w} X \xrightarrow{-u} Y \xrightarrow{-v} Z.$$

T3. For any commutative diagram of the form

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & \Sigma X \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X'
\end{tikzcd}$$

whose rows are triangles, there is $h : Z \to Z'$, not necessarily unique, which makes commutative the diagram:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X'.
\end{tikzcd}$$

T4. *The octahedral axiom:* Any two triangles $X \to Y \to Z \to \Sigma X$ and $Y' \to Y \to Y'' \to \Sigma Y'$ may be completed to a commutative diagram

$$\begin{tikzcd}
 & Y' \arrow[r, equal] \arrow[d] & Y' \arrow[d] & \\
X \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
yX \arrow[r] & Y'' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & \Sigma X \\
 & \Sigma Y' \arrow[r, equal] & \Sigma Y' &
\end{tikzcd}$$

Definition 1.2.2. A functor $\mathcal{T} \to \mathcal{A}$ into an abelian category $\mathcal{A}$ is called *homological* if it sends triangles into exact sequences. A contravariant functor $\mathcal{T} \to \mathcal{A}$ which is homological regarded as a functor $\mathcal{T}^o \to \mathcal{A}$ is called *cohomological* (see [60, Definition 1.1.7 and Remark 1.1.9]). A functor $F : \mathcal{T} \to \mathcal{S}$ between triangulated categories is called *triangulated*, provided that it commutes with shifts, up to a natural isomorphism, and sends triangles in $\mathcal{T}$ into triangles in $\mathcal{S}$.

Examples of (co)homological functors are the covariant (respectively contravariant) hom functors, that is

$$\mathcal{T}(X,-) : \mathcal{T} \to \mathcal{A}b, \text{ repectively } \mathcal{T}(-,X) : \mathcal{T} \to \mathcal{A}b,$$

for every $X \in \mathcal{T}$.

Another important homological functor is the Yoneda embedding $H_{\mathcal{T}} : \mathcal{T} \to \mathrm{mop}(\mathcal{T})$. By the dual of [40, Lemma 2.1] we obtain:

Theorem 1.2.3. *If $\mathcal{T}$ is a triangulated category, then $\mathcal{T}$ has weak cokernels, therefore* $\mathrm{mop}(\mathcal{T})$ *is an abelian category. Moreover for every functor $F : \mathcal{T} \to \mathcal{A}$ into an abelian category, the unique left exact functor $F^* : \mathrm{mop}(\mathcal{T}) \to \mathcal{A}$ extending F is exact if and only if F is homological.*

Note that the above Theorem gives the reason for the following:

Definition 1.2.4. The category $\mathrm{mop}(\mathcal{T})$ (or often the equivalent category $\mathrm{mod}(\mathcal{T})$) is called the *abelianization* of the triangulated category $\mathcal{T}$.

By [60, Corollary 5.1.23], $\mathrm{mop}(\mathcal{T})$ is a Frobenius abelian category, with enough injectives and enough projectives, which are, up to isomorphism, exactly objects of the form $\mathcal{T}(K,-)$ for some $K \in \mathcal{T}$.

If $\mathcal{T}$ and $\mathcal{T}'$ are triangulated categories, and $F : \mathcal{T} \to \mathcal{T}'$ is a triangulated functor, then $H_{\mathcal{T}'} \circ F$ is homological, so it induces a unique, up to isomorphism, exact functor $\hat{F} : \mathrm{mod}(\mathcal{T}) \to \mathrm{mod}(\mathcal{T})'$, such that $H_{\mathcal{T}'} \circ F \cong \hat{F} \circ H_{\mathcal{T}}$. The duality functor $\mathcal{T} \to \mathcal{T}^o$ is (contravariant) triangulated, so it induces as before a unique (contravariant) functor $\mathrm{mod}(\mathcal{T}) \to \mathrm{mod}(\mathcal{T})^o$, which is not hard to see that is a duality. Therefore we obtain:

Lemma 1.2.5. *[39, Corollary 2.11]. If $\mathcal{T}$ is a triangulated category, then there is an equivalence of categories*

$$E : \mathrm{mod}(\mathcal{T}) \to \mathrm{mop}(\mathcal{T}), \textit{ such that } E \circ H \cong H'.$$

Definition 1.2.6. If

$$X_0 \overset{\phi_0}{\to} X_1 \overset{\phi_1}{\to} X_2 \to \dots$$

is a *(direct) tower* of objects and maps in a triangulated category with coproducts $\mathcal{T}$, then its *homotopy colimit* is defined, up to a non-unique isomorphism, by the triangle

$$\coprod_{n\geq 0} X_n \xrightarrow{1-\phi} \coprod_{n\geq 0} X_n \to \underrightarrow{\mathrm{hocolim}}\, X_n \to \Sigma \coprod_{n\geq 0} X_n,$$

where if we denote bu $u_i : X_n \to \coprod_{n\leq 0} X_n$ ($i \geq 0$) the canonical injections, then $\phi u_i = u_{i+1}\phi_i$ for all i. Dually we define the notions *inverse tower* and *homotopy limit.*

1.3 Well–generaterd triangulated categories

Definition 1.3.1. Let $\mathcal{T}$ be a triangulated category. A subcategory $\mathcal{T}'$ is called *triangulated* if $\Sigma\mathcal{T}' = \mathcal{T}'$ and whenever we have a triangle $X \to Y \to Z \to \Sigma X$ in $\mathcal{T}$ with $X, Y \in \mathcal{T}$' then $Y \in \mathcal{T}'$.

Definition 1.3.2. A cardinal κ is said to be *regular* provided that it is infinite and it can not be written as a sum of less than κ cardinals, all smaller than κ.

Definition 1.3.3. Let $\mathcal{T}$ be a triangulated category with coproducts. For regular cardinal λ, a λ*–localizing* subcategory of $\mathcal{T}$ is a triangulated subcategory closed under λ–coproducts. A *localizing* subcategory is a subcategory which is λ–localizing, for all λ. Consequently a full triangulated subcategory $\mathcal{L}$ of $\mathcal{T}$ is called localizing exactly if it is closed under taking coproducts in $\mathcal{T}$. Given a class of objects $\mathcal{S} \subseteq \mathcal{T}$, we denote by $\mathrm{Loc}(\mathcal{S})$ the smallest localizing subcategory of $\mathcal{T}$ containing $\mathcal{S}$. Dually are defined *colocalizing* subcategories.

Given an arbitrary full subcategory $\mathcal{S} \subseteq \mathcal{T}$, we denote

$$
\begin{aligned}
{}^{\perp}\mathcal{S} &= \{X \in \mathcal{T} \mid \mathcal{T}(X, S) = 0 \text{ for all } S \in \mathcal{S}\} \\
\mathcal{S}^{\perp} &= \{X \in \mathcal{T} \mid \mathcal{T}(S, X) = 0 \text{ for all } S \in \mathcal{S}\},
\end{aligned}
$$

We say that $\mathcal{S} \subseteq \mathcal{T}$ is Σ*-stable* if it is closed under suspensions and desuspensions, that is $\Sigma\mathcal{S} \subseteq \mathcal{S}$ and $\Sigma^{-1}\mathcal{S} \subseteq \mathcal{S}$. Note that if $\mathcal{S}$ a Σ-stable subcategory of $\mathcal{T}$, then ${}^{\perp}\mathcal{S}$ and $\mathcal{S}^{\perp}$ are triangulated subcategories.

Definition 1.3.4. Let $\mathcal{T}$ is a triangulated category with coproducts. Consider a set of objects $\mathcal{S} \subseteq \mathcal{T}$ which is Σ-stable. We say that $\mathcal{T}$ is generated (in the triangulated sense) by $\mathcal{S}$, provided that an object $T \in \mathcal{T}$ vanishes, whenever $\mathcal{T}(S, T) = 0$ for all $S \in \mathcal{S}$. Further we say that $\mathcal{T}$ is perfectly generated by the set of objects $\mathcal{S}$ if $\mathcal{S}$ generates $\mathcal{T}$ and, for any $S \in \mathcal{S}$, the map $\mathcal{T}(S, \coprod_{i\in I} X_i) \to \mathcal{T}(S, \coprod_{i\in I} Y_i)$ is surjective, for every set of maps $\{X_i \to Y_i \mid i \in I\}$ such that $\mathcal{T}(S, X_i) \to \mathcal{T}(S, Y_i)$ is surjective, for all $i \in I$. Finally $\mathcal{T}$ is called λ*-compactly generated*, where λ is a regular cardinal, provided that $\mathcal{T}$ is perfectly generated by a set of objects which are also λ*–small*, that is, every map $S \to \coprod_{i\in I} X_i$, with $S \in \mathcal{S}$, factors trough a coproduct $\coprod_{i\in I'} X_i$ with $\operatorname{card} I' < \lambda$; the category $\mathcal{T}$ is *well–generated* if it is λ-compactly generated, for some λ. A $\aleph_0$-compactly generated triangulated category is also called simply *compactly generated*.

Following [42, Theorem A], this definition is equivalent to the original one given by Neeman, modulo the assumption that the isomorphism classes of λ–compact objects form a set. Note that, by Corollary 3.2.11, if $\mathcal{T}$ is perfectly generated by $\mathcal{S}$, then $\mathcal{T}$ coincides with its smallest $\aleph_1$–localizing subcategory which contains arbitrary coproducts of objects in $\mathcal{S}$.

Let $\mathcal{T}$ be triangulated category which is κ-compactly generated by a set $\mathcal{S}$. For any $\lambda \geq \kappa$ we consider the subcategory of λ–compact objects, that is the smallest λ localizing subcategory of $\mathcal{T}$ which contains $\mathcal{S}$ and denote it by $\mathcal{T}^{\lambda}$. The objects in $\mathcal{T}^{\lambda}$ are called λ*–compact*. Clearly it is essentially small and a skeleton of $\mathcal{T}^{\lambda}$ generates $\mathcal{T}$. Moreover this subcategory is independent of $\mathcal{S}$.

1.4 Brown representability

Let $\mathcal{T}$ be a triangulated category with coproducts. Then for all $X \in \mathcal{T}$ the contravariant hom functor $\mathcal{T}(-,X) : \mathcal{T} \to \mathcal{A}b$ is cohomological and sends coproducts into products. The converse of this property is called Brown representability, namely:

Definition 1.4.1. We say that $\mathcal{T}$ satisfies *Brown representability* if it has coproducts and every cohomological functor $F : \mathcal{T} \to \mathcal{A}b$ which sends coproducts into products is (contravariantly) representable, that is, it is naturally isomorphic to $\mathcal{T}(-,X)$ for some $X \in \mathcal{T}$.

Note that the dual $\mathcal{T}^o$ satisfies Brown representability exactly if $\mathcal{T}$ has products and every homological product preserving functor $F : \mathcal{T} \to \mathcal{A}b$ is (covariantly) representable, that is it is isomorphic to $\mathcal{T}(X,-)$ for some $X \in \mathcal{T}$.

Theorem 1.4.2. *[60, Theorem 8.3.3] and [60, theorem 8.6.1]. Well–generated triangulated categories satisfy Brown representability. Compactly generated triangulated categories satisfies Brown representability for the dual too.*

Triangulated categories are usually not complete nor cocomplete. Therefore usual techniques for constructing adjoints are useless. The following property which is a direct consequence of Brown representability provides a way to overcome this lack:

Theorem 1.4.3. *[60, Theorem 8.4.4] or [44, Theorem 5.1.1]. Let $\mathcal{T}$ and $\mathcal{S}$ be a triangulated categories.*

(1) If $\mathcal{T}$ satisfies Brown representability then a triangulated functor $F : \mathcal{T} \to \mathcal{S}$ has a right adjoint if and only if F preserves coproducts.

(2) If $\mathcal{T}^o$ satisfies Brown representability then a triangulated functor $F : \mathcal{T} \to \mathcal{S}$ has a left adjoint if and only if F preserves products.

Proof. Since (1) and (2) are dual, it is enough to prove (1). Let $F : \mathcal{T} \to \mathcal{T}'$ be a triangulated functor. If it has a right adjoint, then it preserves coproducts by the general theory of adjoint functors. Conversely if it preserves coproducts, then for all $Y \in \mathcal{S}$ the functor $\mathcal{S}(F(-),Y) : \mathcal{T} \to \mathcal{A}b$ is cohomological and sends coproducts into products. Therefore Brown representability for T gives us an object $X \in \mathcal{T}$ such that $\mathcal{S}(F(-),Y) \cong \mathcal{T}(,X)$ naturally. This object X is unique, up to a natural isomorphism, by Yoneda Lemma. Thus the assignment $Y \mapsto X$ induces a functor $\mathcal{S} \to \mathcal{T}$ which is the left adjoint of F. □

Another interesting feature of triangulated categories satisfying Brown representability is the existence of the products:

Theorem 1.4.4. *[60, Proposition 8.4.6]. If the triangulated category $\mathcal{T}$ satisfies Brown representability, then it has products. Dually if $\mathcal{T}^o$ satisfies Brown representability, then $\mathcal{T}$ has coproducts.*

Proof. Let $X_i \in \mathcal{T}_{i\in I}$ be a family of objects indexed by an arbitrary set I. The functor $\prod_{i\in I}\mathcal{T}(-,X_i)$ is cohomological and sends coproducts into products, therefore it is representable. An object $X \in T$ representing it has to be the product of the considered family. □

1.5 The homotopy category and the derived category

Throughout this section $\mathcal{A}$, will denote an additive category. We suppose also that $\mathcal{A}$ has split idempotents.

Definition 1.5.1. The *the category of complexes* over $\mathcal{A}$ which is denoted by $\mathbf{C}(\mathcal{A})$ has as objects chains of objects and morphisms (these morphisms are called *differentials*) in $\mathcal{A}$ of the form

$$X^\bullet = \cdots \to X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \to \cdots,$$

such that $d_X^n d_X^{n-1} = 0$ for all $n \in \mathbb{Z}$. The morphisms in the category $\mathbf{C}(\mathcal{A})$ are families $f^\bullet = (f^n)_{n\in\mathbb{Z}}$ of morphisms in $\mathcal{A}$ commuting with differentials.

Limits and colimits in the category $\mathbf{C}(\mathcal{A})$ are computed component–wise, provided that the respective constructions may be performed in $\mathcal{A}$. In particular $\mathbf{C}(\mathcal{A})$ is abelian (Grothendieck) if $\mathcal{A}$ is so.

Definition 1.5.2. The *homotopy category* of complexes $\mathbf{K}(\mathcal{A})$ has the same objects as $\mathbf{C}(\mathcal{A})$ and the morphism space is defined by

$$\mathbf{K}(\mathcal{A})(X^\bullet, Y^\bullet) = \mathbf{C}(\mathcal{A})(X^\bullet, Y^\bullet)/_\sim$$

where $\sim$ is an equivalence relation called *homotopy*, defined as follows: two maps of complexes $(f^n)_{n\in\mathbb{Z}}, (g^n)_{n\in\mathbb{Z}} : X \to Y$ are homotopically equivalent if there is $s^n : X^n \to Y^{n-1}$, for all $n \in \mathbb{Z}$ such that $f^n - g^n = d_Y^{n-1}s^n + s^{n+1}d_X^n$.

Note that $\mathbf{C}(\mathcal{A})$ is an exact category (in the sense of [35, Section 4]) with respect to all short exact sequences which split in each degree (see [35, Example 4.3]), and $\mathbf{K}(\mathcal{A})$ may be constructed as the stable category of this exact category by [35, Example 6.1]. Hence $\mathbf{K}(\mathcal{A})$ is a triangulated category. Note that the structure of triangulated category comes with a translation functor denoted by Σ, where $(\Sigma X)^n = X^{n+1}$ and $d_{\Sigma X}^n = -d_X^{n+1}$.

In the rest of the section, assume $\mathcal{A}$ is abelian. For a complex $X^\bullet \in \mathbf{C}(\mathcal{A})$ and an $n \in \mathbb{Z}$, denote $\mathrm{Z}^n(X^\bullet) = \ker d^n$ and $\mathrm{B}^n(X^\bullet) = \operatorname{im} d^{n-1}$, and call them the object of *n-th cocycles* and the object of n-th *boundaries* respectively. It is clear that $\mathrm{B}^n(X^\bullet) \le \mathrm{Z}^n(X^\bullet) \le X^n$, thus we are allowed to consider $\mathrm{H}^n(X^\bullet) = \mathrm{Z}^n(X^\bullet)/\mathrm{B}^n(X^\bullet)$, the *n-th cohomology* of $X^\bullet$. A complex $X^\bullet$ is called *acyclic* if $\mathrm{H}^n(X^\bullet) = 0$ for all $n \in \mathbb{Z}$.

Definition 1.5.3. A map $f^\bullet : X^\bullet \to Y^\bullet$ in $\mathbf{K}(\mathcal{A})$ which induces isomorphisms in cohomology is called *quasi–isomorphism*. The *derived category* $\mathbf{D}(\mathcal{A})$ is the category of fractions of $\mathbf{K}(\mathcal{A})$ with respect to all quasi–isomorphisms.

Note that two complexes $X^\bullet$ and $Y^\bullet$ are isomorphic in $\mathbf{K}(\mathcal{A})$ if there are maps of complexes $f^\bullet : X^\bullet \to Y^\bullet$ and $g^\bullet : Y^\bullet \to X^\bullet$ such that both compositions $g^\bullet f^\bullet$ and $f^\bullet g^\bullet$ are homotopically equivalent to the respective identities. If this is the case, it is not hard to see that $X^\bullet$ and $Y^\bullet$ have the same cohomology, so the functors $\mathrm{H}^n : \mathbf{C}(\mathcal{A}) \to \mathcal{A}$, $n \in \mathbb{Z}$, induce well defined functors $\mathbf{K}(\mathcal{A}) \to \mathcal{A}$. Therefore the full subcategory of acyclic complexes is a triangulated subcategory of $\mathbf{K}(\mathcal{A})$. Then $\mathbf{D}(\mathcal{A})$ can be obtained as as the Verdier quotient (see [60, Section 2.1]) of $\mathbf{K}(\mathcal{A})$ modulo the triangulated subcategory of all acyclic complexes. It follows $\mathbf{D}(\mathcal{A})$ is triangulated too.

A priori there is no reason to expect that $\mathbf{D}(\mathcal{A})$ has small hom–sets, because this is the case of a Verdier quotient in general. However we will only consider derived categories for which, at some point, we are able to show that they have small sets.

We will see every object of $\mathcal{A}$ as a complex concentrated in degree zero, providing embeddings of $\mathcal{A}$ in any of the categories $\mathbf{C}(\mathcal{A})$, $\mathbf{K}(\mathcal{A})$ or $\mathbf{D}(\mathcal{A})$. Note also that, if $\mathcal{A}$ has (co)products then $\mathbf{C}(\mathcal{A})$ and $\mathbf{K}(\mathcal{A})$ have (co)products and the canonical functor $\mathbf{C}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ preserves them. If, in addition, these (co)products are exact then the full subcategory of acyclic complexes is closed under (co)products, therefore $\mathbf{D}(\mathcal{A})$ has also (co)products and the quotient functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ preserves them, by [44, Theorem 3.5.1]. Note that, the exactness of (co)products in $\mathcal{A}$ is only a sufficient condition, and not a necessary one, for the existence of (co)products in $\mathbf{D}(\mathcal{A})$.

Chapter 2

Abelianization

In this chapter we reformulate Brown representability at the level of the abelianization of the considered triangulated category and we give a new proof to a representability criterion due to Heller. The results presented here were first published in [9] and [51].

2.1 A reformulation of Brown representability

We call an abelian category $\mathcal{A}$ *admissible* if it is AB3 and has enough injectives. It is well-known that such a category must be also AB4.

Theorem 2.1.1. *The following are equivalent, for a triangulated category with arbitrary coproducts $\mathcal{T}$:*

(i) $\mathcal{T}$ satisfies the Brown representability theorem.

(ii) For every homological, coproducts preserving functor $f : \mathcal{T} \to \mathcal{A}$, into an admissible abelian category $\mathcal{A}$, the induced functor

$$f_* : \mathrm{mod}(\mathcal{T}) \to \mathcal{A}$$

has a right adjoint.

(iii) Every exact, coproducts preserving functor $F : \mathrm{mod}(\mathcal{T}) \to \mathcal{A}$, into an admissible, abelian category $\mathcal{A}$, has a right adjoint.

(iv) Every exact, coproducts preserving functor $F : \mathrm{mod}(\mathcal{T}) \to \mathcal{A}b^o$ has a right adjoint.

Proof. (i)⇒(ii). Let $f : \mathcal{T} \to \mathcal{A}$ be a homological functor into an abelian, admissible category $\mathcal{A}$. Let $I \in \mathcal{A}$ be an injective object. Then the functor

$$\mathcal{A}(f(-), I) = \mathcal{A}(-, I) \circ f : \mathcal{T} \to \mathcal{A}b$$

is cohomological, and sends coproducts into products. Then it is representable, by Brown representability theorem; so there is a unique, up to a natural isomorphism, $x_I \in \mathcal{T}$, such that $\mathcal{A}(f(-), I) \cong \mathcal{T}(-, x_I)$. Since $\mathcal{A}$ has enough injectives, the assignment $I \mapsto H(x_I)$ defines a unique, up to isomorphism, left exact functor $G : \mathcal{A} \to \text{mod}(\mathcal{T})$. It is directly verified that G is the right adjoint of f_*.

(ii)⇒(iii) is obvious, since, under the assumptions of (iii), we have

$$F \cong (F \circ H)_*.$$

(iii)⇒(iv) follows immediately since $\mathcal{A}b^o$ is admissible.

(vi)⇒(i). Let $f : \mathcal{T} \to \mathcal{A}b$ be a cohomological functor, sending coproducts into products. Then the functor

$$F : \text{mod}(\mathcal{T}) \to \mathcal{A}b^o, \;\; F(X) = (\text{mod}(\mathcal{T}))(X, f)$$

is exact, coproducts preserving (actually F is the composition of f_* with the duality functor of $\mathcal{A}b$). By hypothesis, F has a right adjoint $G : \mathcal{A}b^o \to \text{mod}(\mathcal{T})$. We deduce $F(X) \cong (\text{mod}(\mathcal{T}))(X, G(\mathbb{Z}))$. Further $f \cong G(\mathbb{Z})$, showing that $f \in \text{mod}(\mathcal{T})$ and f has to be injective, since it represents the exact functor F. Therefore, $f \cong \mathcal{T}(-, x)$, for some $x \in \mathcal{T}$. □

We record also the dual of the preceding results (for this, we will say that the abelian category $\mathcal{A}$ is *co-admissible* if $\mathcal{A}^o$ is admissible):

Theorem 2.1.2. *The following are equivalent, for a triangulated category with arbitrary products $\mathcal{T}$:*

(i) $\mathcal{T}^o$ satisfies the Brown representability theorem.

(ii) For every homological, products preserving functor $f : \mathcal{T} \to \mathcal{A}$, into a co-admissible, abelian category $\mathcal{A}$, the induced functor

$$f^* : \text{mop}(\mathcal{T}) \to \mathcal{A}$$

has a left adjoint.

(iii) Every exact, products preserving functor $F : \text{mop}(\mathcal{T}) \to \mathcal{A}$, into a co-admissible, abelian category $\mathcal{A}$, has a left adjoint.

(iv) Every exact, products preserving functor $F : \text{mop}(\mathcal{T}) \to \mathcal{A}b$ has a left adjoint.

Remark 2.1.3. According to Lemma 1.2.5, for any triangulated category $\mathcal{T}$, we have an equivalence of categories $E : \text{mod}(\mathcal{T}) \to \text{mop}(\mathcal{T})$, such that $E \circ H = H'$, we may freely interchange $\text{mod}(\mathcal{T})$ and $\text{mop}(\mathcal{T})$ in Theorems 2.1.1 and 2.1.2.

Remark 2.1.4. Consider a triangulated category, with arbitrary coproducts (products) $\mathcal{T}$. Theorems 2.1.1 and 2.1.2 provide reformulations of the Brown representability theorem for $\mathcal{T}$ respectively $\mathcal{T}^o$ in terms of abelian category $\text{mod}(\mathcal{T})$.

2.2 Heller's criterion revisited

In this section we consider a triangulated $\mathcal{T}$ with split idempotents. Recall that that $\mathcal{T}$ has split idempotents, provided that $\mathcal{T}$ has countable coproducts or products, according to [60, Proposition 1.6.8] or its dual.

Let $F : \mathcal{T} \to \mathcal{A}b$ be a chomological functor, and consider $F^* : \mathrm{mop}(\mathcal{T}) \to \mathcal{A}b$ the unique exact functor extending F (see 1.2.3). Observe that in this particular case when the codomain of F is the category $\mathcal{A}b$ of all abelian groups, then it can be easily seen that $F^*(X) \cong \mathrm{Hom}_{\mathcal{T}^o}(X, F)$, naturally for all $X \in \mathrm{mop}(\mathcal{T})$. Thus we obtain:

Lemma 2.2.1. *If $\mathcal{T}$ is a triangulated category with split idempotents, then a homological functor $F : \mathcal{T} \to \mathcal{A}b$ is representable if and only if its extension $F^* : \mathrm{mop}(\mathcal{T}) \to \mathcal{A}b$ is representable.*

Proof. As before $F^*(X) \cong \mathrm{Hom}_{\mathcal{T}^o}(X, F)$, for all $X \in \mathrm{mop}(\mathcal{T})$. If F is representable, then $F \in \mathrm{mop}(\mathcal{T})$, so F^* is represented by F. Conversely if F^* is representable by an object in $\mathrm{mop}(\mathcal{T})$ then this object must be isomorphic to F, therefore $F \in \mathrm{mop}(\mathcal{T})$. Because F^* is exact, F must be projective, hence representable (see [60, Lemma 5.1.11]). □

Lemma 2.2.2. *If $\mathcal{T}$ is a triangulated category with split idempotents, then a cohomological functor $F : \mathcal{T} \to \mathcal{A}b$ has a solution object if and only if $F^* : \mathrm{mop}(\mathcal{T}) \to \mathcal{A}b$ has a solution object.*

Proof. Suppose F has a solution object, i.e. there is a functorial epimorphism $H'(K) = \mathcal{T}(K, -) \to F \to 0$, with $K \in \mathcal{T}$. In order to show that F^* has a solution object, it is enough to prove that the induced natural transformation

$$\mathrm{Hom}_{\mathcal{T}^o}(-, H'(K)) \to \mathrm{Hom}_{\mathcal{T}^o}(-, F) \cong F^*$$

is an epimorphism. That is, we want to show that the map

$$\mathrm{Hom}_{\mathcal{T}^o}(X, H'(K)) \to \mathrm{Hom}_{\mathcal{T}^o}(X, F)$$

is surjective, for all $X \in \mathrm{mop}(\mathcal{T})$. According to [60, 5.1.23] every finitely presentable $\mathcal{T}^o$–module X admits an embedding $0 \to X \to \mathcal{T}(U, -)$, (in mod $(\mathcal{T}^o)$, that is an epimorphism from the projective

$$H'(U) \to X \to 0,$$

starting from the projective object $H'(U)$ in the opposite category $\mathrm{mop}(\mathcal{T})$. Since $H'(K) \in \mathrm{mop}(\mathcal{T})$ is projective–injective and F^* is exact, we obtain a diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{T}^o}(H'(U), H'(K)) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{T}^o}(X, H'(K)) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}_{\mathcal{T}^o}(H'(U), F) \arrow[r] & \mathrm{Hom}_{\mathcal{T}^o}(X, F) \arrow[r] & 0
\end{tikzcd}.$$

By Yoneda lemma we know that the first vertical map is isomorphic to

$$\mathcal{T}(K,U) \to F(U),$$

hence it is surjective, thus the diagram above proves the direct implication.

Conversely if there is $X \in \mathrm{mop}(\mathcal{T})$ and a natural epimorphism

$$\mathrm{Hom}_{\mathcal{T}^o}(-,X) \to \mathrm{Hom}_{\mathcal{T}^o}(-,F) \to 0,$$

then let $H'(K) \to X \to 0$ be an epimorphism in $\mathrm{mod}(\mathcal{T}^o)$ (that is a monomorphism in the opposite direction in $\mathrm{mop}(\mathcal{T})$), with $K \in \mathcal{T}$. Consider the composed map

$$\mathrm{Hom}_{\mathcal{T}^o}(-,H'(K)) \to \mathrm{Hom}_{\mathcal{T}^o}(-,X) \to \mathrm{Hom}_{\mathcal{T}^o}(-,F).$$

Evaluating it at $H'(U)$ for an arbitrary $U \in \mathcal{T}$, we obtain a surjective natural map $\mathcal{T}(K,U) \to F(U)$, hence F has a solution object. □

The next Theorem was shown by Heller in [28, Theorem 1.4], hence we call it *Heller's criterion* of representability. However, our argument is different of Heller's proof, and it seems to be more conceptual, because the conclusion follows from the celebrated Freyd's Adjoint Functor Theorem.

Theorem 2.2.3. *If $\mathcal{T}$ is a triangulated category with products, then a homological product preserving functor $F : \mathcal{T} \to \mathcal{A}b$ is representable if and only if it has a solution object.*

Proof. Under the hypotheses imposed on $\mathcal{T}$ and F, the abelian category $\mathrm{mop}(\mathcal{T})$ is complete and the induced functor $F^* : \mathrm{mop}(\mathcal{T}) \to \mathcal{A}b$ preserves limits. Therefore it is representable if and only if it has a solution object, by Freyd's Adjoint Functor Theorem. Thus the conclusion follows by combining Lemmas 2.2.1 and 2.2.2. □

Remark 2.2.4. Theorem 2.2.3 says more than the Neeman's Freyd style representability theorem [61, Theorem 1.3]. Indeed the cited result states that if every cohomological functor which sends coproducts into products has a solution objects, then every such a functor is representable, whereas Theorem 2.2.3 involves a fixed functor.

In a particular case, namely in the presence of products, we can derive from the results above the dual of [63, Proposition 1.4]. In order to state this, recall that if $\mathcal{T}$ is a full subcategory of $\mathcal{T}$ then a $\mathcal{T}$*–preenvelope* of $T \in \mathcal{T}$ is a morphism $T \to X_T$ with $X_T \in \mathcal{T}$ such that the induced map $\mathcal{T}(X_T,X) \to \mathcal{T}(T,X)$ is surjective for all $X \in \mathcal{T}$. Dually we define the concept of *precover*. The subcategory $\mathcal{T}$ is called *preenveloping* is every object in $\mathcal{T}$ admits a $\mathcal{T}$-preenvelope.

Corollary 2.2.5. *Let $\mathcal{T}'$ be a triangulated category with products, and let $\mathcal{T}$ be a colocalizing subcategory. The following are equivalent:*

(i) The inclusion $\mathcal{T} \to \mathcal{T}'$ has a left adjoint.

(ii) Every object in $\mathcal{T}'$ admits a $\mathcal{T}$–preenvelope.

Proof. Since the implication (i)⇒(ii) follows from the general theory of adjoint functors, we only need to show the converse. But this follows immediately from Theorem 2.2.3 since, if $I : \mathcal{T} \to \mathcal{T}'$ is the inclusion functor, then for every $T \in \mathcal{T}'$ the functor $\mathcal{T}'(T, I(-)) : \mathcal{T} \to \mathcal{A}b$ is homological, preserves products and has a solution object, given by the functorial epimorphism $\mathcal{T}(X_T, -) \to \mathcal{T}'(T, I(-))$, where $T \to X_T$ is a $\mathcal{T}$–preenvelope of X. $\square$

Chapter 3

Deconstructibility in triangulated categories

In this chapter we define deconstructible triangulated categories and we show that they satisfies Brown representability. One important feature of our approach is that it can be dualized. Further we show that well–generated triangulated categories are deconstructible, therefore we find a new proof of the fact that they satisfy Brown representability. The material is essentialy taken from [51] and [49], but some changes are also operated. Namely, in the same sense in which our Theorem 2.2.3 is an improvement of Neeman's [61, Theorem 1.3], we improve the main result of [49], more precisely [49, Theorem 3.7], which originally uses the Neeman's result, in order to obtain Theorem 3.3.3.

3.1 Deconstructibiliy

Consider a Σ-closed set of objects in $\mathcal{T}$ and denote it by $\mathcal{S}$. We define $\mathrm{Prod}(\mathcal{S})$ to be the full subcategory of $\mathcal{T}$ consisting of all direct factors of products of objects in $\mathcal{S}$. Next we define inductively $\mathrm{Prod}_0(\mathcal{S}) = \{0\}$ and $\mathrm{Prod}_n(\mathcal{S})$ is the full subcategory of $\mathcal{T}$ which consists of all objects Y lying in a triangle

$$X \to Y \to Z \to \Sigma X$$

with $X \in \mathrm{Prod}(\mathcal{S})$ and $Z \in \mathrm{Prod}_n(\mathcal{S})$. Clearly $\mathrm{Prod}_1(\mathcal{S}) = \mathrm{Prod}(\mathcal{S})$ and the construction leads to an ascending chain $\mathrm{Prod}_0(\mathcal{S}) \subseteq \mathrm{Prod}_1(\mathcal{S}) \subseteq \cdots$. Recall that $\mathcal{S}$ is Σ-closed, hence the same is true for $\mathrm{Prod}_n(\mathcal{S})$, by [61, Remark 07]. The same [61, Remark 07] says, in addition, that if $X \to Y \to Z \to \Sigma X$ is a triangle with $X \in \mathrm{Prod}_n(\mathcal{S})$ and $\mathrm{Prod}_m(\mathcal{S})$ then $Z \in \mathrm{Prod}_{n+m}(\mathcal{S})$. An object $X \in \mathcal{T}$ will be called *$\mathcal{S}$-cofiltered* if it may be written as a homotopy limit $X \cong \underleftarrow{\mathrm{holim}}\, X_n$ of an inverse tower

$$X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$$

with $X_0 \in \mathrm{Prod}_0(\mathcal{S})$, and X_{n+1} lying in a triangle $P_n \to X_{n+1} \to X_n \to \Sigma P_n$, for some $P_n \in \mathrm{Prod}_1(\mathcal{S})$. Inductively we have $X_n \in \mathrm{Prod}_n(\mathcal{S})$, for all $n \in \mathbb{N}^*$. The dual notion must be surely called *filtered*, and the terminology comes from the analogy with the filtered objects in an abelian category (see [27, Definition 3.1.1]). Further the same analogy leads to the folllowing:

Definition 3.1.1. We say that $\mathcal{T}$ (respectively, $\mathcal{T}^o$) is *deconstructible* if $\mathcal{T}$ has coproducts (products) and there is a Σ-closed set $\mathcal{S} \subseteq \mathcal{T}$, which is not a proper class, such that every object $X \in \mathcal{T}$ is $\mathcal{S}$-filtered (cofiltered).

Note that we may define deconstructibility without closure under suspensions and desuspension. Indeed if every $X \in \mathcal{T}$ is $\mathcal{S}$–(co)filtered, then it is also $\overline{\mathcal{S}}$–(co)filtered, where $\overline{\mathcal{S}}$ is the closure of $\mathcal{S}$ under suspensions and desuspensions.

Lemma 3.1.2. *Let $\mathcal{T}$ be a triangulated category with products. If $\mathcal{T}^o$ is deconstructible, then every homological product preserving functor $F : \mathcal{T} \to \mathcal{A}b$ has a solution object.*

Proof. We shall prove a statement equivalent to the conclusion, namely that the category of elements $\mathcal{T}/F$ has a weakly initial object.

Let $\mathcal{S} \subseteq \mathcal{T}$ be a Σ-closed set of objects, such that every object of $\mathcal{T}$ is $\mathcal{S}$–cofiltered. By [61, Lemma 2.3], we know that the category $\mathrm{Prod}_n(\mathcal{S})/F$ has a weakly initial object denoted (T_n, t_n), for all $n \in \mathbb{N}$. Let I be the non–empty set of all inverse towers of the form

$$T_0 \xleftarrow{w_0} T_1 \xleftarrow{w_1} T_2 \longleftarrow \cdots$$

with $F(w_n)(t_{n+1}) = t_n$, for all $n \in \mathbb{N}$, and denote by $T(i)$ the homotopy limit of the tower $i \in I$. By [6, Lemma 5.8(2)], there is an exact sequence

$$0 \to \varprojlim{}^{(1)} F(T_n[-1]) \to F(\underleftarrow{\mathrm{holim}}\, T_n) \to \varprojlim F(T_n) \to 0.$$

Clearly $(t_n)_{n\in\mathbb{N}} \in \varprojlim F(T_n)$, thus there exists

$$t(i) \in F(T(i)) = F(\underleftarrow{\mathrm{holim}}\, T_n)$$

which maps in $(t_n)_{n\in\mathbb{N}}$ via the surjective morphism above. We claim that (T, t) is a weakly initial object in $\mathcal{T}/F$, where $T = \prod_{i\in I} T(i)$ and $t = (t(i))_{i\in I}$. In order to prove the claim, consider an object $X \in \mathcal{T}$. By hypothesis, there is an inverse tower

$$X_0 \xleftarrow{u_0} X_1 \xleftarrow{u_1} X_2 \longleftarrow \cdots$$

whose homotopy limit is X such that $X_0 = 0 \in \mathrm{Prod}_0(\mathcal{S})$, and every X_{n+1} lies in a triangle $P_n \to X_{n+1} \xrightarrow{u_n} X_n \to \Sigma P_n$, for some $P_n \in \mathrm{Prod}(\mathcal{S})$. We use again [6, Lemma 5.8(2)] for constructing the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim^{(1)} \mathcal{T}(T, X_n[-1]) \arrow[r] \arrow[d] & \mathcal{T}(T, \underleftarrow{\mathrm{holim}}\, X_n) \arrow[r] \arrow[d] & \varprojlim \mathcal{T}(T, X_n) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varprojlim^{(1)} F(X_n[-1]) \arrow[r] & F(\underleftarrow{\mathrm{holim}}\, X_n) \arrow[r] & \varprojlim F(X_n) \arrow[r] & 0
\end{tikzcd}$$

whose columns are induced by the natural transformations which correspond to $t \in F(T)$ under Yoneda Lemma. If we show that the two extreme vertical arrows are surjective, the same is true for the middle arrow too, and we are done. But for the first vertical map this follows by the commutative diagram:

$$\begin{tikzcd} \prod_{n\in\mathbb{N}} \mathcal{T}(T, X_n[-1]) \arrow[r] \arrow[d] & \varprojlim^{(1)} \mathcal{T}(T, X_n[-1]) \arrow[d] \\ \prod_{n\in\mathbb{N}} F(X_n[-1]) \arrow[r] & \varprojlim^{(1)} F(X_n[-1]) \end{tikzcd}$$

whose arrows connected with the south-west corner are both surjective.

In order to prove that the third vertical map above is surjective, we consider an element $x \in \varprojlim F(X_n)$, that is $x = (x_n)_{n\in\mathbb{N}} \in \prod F(X_n)$ such that $F(u_n)(x_{n+1}) = x_n$, for all $n \in \mathbb{N}$. Next we construct a commutative diagram

$$\begin{tikzcd} T_0 \arrow[d, "f_0"'] & T_1 \arrow[l, "w_0"'] \arrow[d, "f_1"'] & T_2 \arrow[l, "w_1"'] \arrow[d, "f_2"] & \cdots \arrow[l] \\ X_0 & X_1 \arrow[l, "u_0"] & X_2 \arrow[l, "u_1"] & \cdots \arrow[l] \end{tikzcd}$$

whose upper line is a tower in I, and satisfying $F(f_n)(t_n) = x_n$ for all $n \in \mathbb{N}$. This construction is performed inductively as follows: f_0 comes from the fact that $(T_0, t_0) = (0, 0)$ is weakly initial in $\mathrm{Prod}_0(\mathcal{S})/F$. Suppose the first n steps are done. We construct the following commutative diagram whose rows are triangles and the middle square is homotopy pull-back (see [60, Definition 1.4.1]):

$$\begin{tikzcd} P_n \arrow[r] \arrow[d, equal] & Y_{n+1} \arrow[r] \arrow[d] & T_n \arrow[r] \arrow[d, "f_n"] & \Sigma P_n \arrow[d, equal] \\ P_n \arrow[r] & X_{n+1} \arrow[r, "u_n"] & X_n \arrow[r] & \Sigma P_n \end{tikzcd}$$

The upper triangle shows that $Y_{n+1} \in \mathrm{Prod}_{n+1}(\mathcal{S})$ where (T_{n+1}, t_{n+1}) is weakly initial, hence we find a map $(T_{n+1}, t_{n+1}) \to (Y_{n+1}, y_{n+1})$ in $\mathrm{Prod}_{n+1}(\mathcal{S})/F$. Now Y_{n+1} is obtained via a triangle

$$Y_{n+1} \to T_n \times X_{n+1} \stackrel{(f_n, -u_n)}{\longrightarrow} X_n \to \Sigma Y_{n+1}.$$

Applying the homological functor F we get an exact sequence:

$$F(Y_{n+1}) \to F(T_n) \times F(X_{n+1}) \stackrel{(F(f_n), -F(u_n))}{\longrightarrow} F(X_n).$$

Since $F(f_n)(t_n) - F(u_n)(x_{n+1}) = x_n - x_n = 0$ we get an element $y_{n+1} \in Y_{n+1}$, which maps in (t_n, x_{n+1}), via the first morphism of the exact sequence above. The morphism f_{n+1} is the composition $T_{n+1} \to Y_{n+1} \to X_{n+1}$. The upper row above is, as noticed, an inverse tower in I, and let denote it by i. Finally the element $t \in T$ maps to $(x_n)_{n\in\mathbb{N}} \in \varprojlim F(X_n)$, via the map $F(T) \to F(T(i)) \to \varprojlim F(T_n) \to \varprojlim F(X_n)$, proving that the map $\varprojlim \mathcal{T}(T, X_n) \to \varprojlim F(X_n)$ is surjective. $\square$

Combining Theorem 2.2.3 and Lemma 3.1.2 we obtain:

Theorem 3.1.3. *Let $\mathcal{T}$ be a triangulated category with products. If $\mathcal{T}^o$ is deconstructible, then $\mathcal{T}^o$ satisfies Brown representability.*

The above Theorem is a useful criterion for a triangulated category to satisfy Brown representability, as we will see in the next chapters. We call it the *deconstructibility criterion.* Note that as all considerations in this section, deconstructibility criterion may be immediately dualized.

3.2 Projective classes

The following Lemma is the dual of [6, Lemma 5.8 (2)]. Note that we give a slightly more general version, replacing the category $\mathcal{A}b$ (more precisely $\mathcal{A}b^o$) with an abelian AB4 category $\mathcal{A}$, where the derived functors $\varinjlim^{(i)}$ of the colimits are computed in the usual manner, by using homology of a complex. Moreover, [6, Lemma 5.8 (1)] is a direct consequence of this dual, together with the exactness of colimits in $\mathcal{A}b$ (that is $\varinjlim^{(1)} = 0$).

Lemma 3.2.1. *Consider a tower $x_0 \overset{\phi_0}{\to} x_1 \overset{\phi_1}{\to} x_2 \overset{\phi_2}{\to} x_3 \to \cdots$ in $\mathcal{T}$. If $F : \mathcal{T} \to \mathcal{A}$ a homological functor which preserves countable coproducts into an abelian AB4 category $\mathcal{A}$, then we have a Milnor exact sequence*

$$0 \to \varinjlim F(x_n) \to F(\underrightarrow{\mathrm{hocolim}}\, x_n) \to \varinjlim{}^{(1)} F(\Sigma x_n) \to 0$$

and $\varinjlim^{(i)} F(x_n) = 0$ for $i \geq 2$.

Corollary 3.2.2. *Consider a tower $x_0 \overset{\phi_0}{\to} x_1 \overset{\phi_1}{\to} x_2 \overset{\phi_2}{\to} x_3 \to \cdots$ in $\mathcal{T}$. If $F : \mathcal{T} \to \mathcal{A}$ is a homological functor, which preserves countable coproducts into an abelian AB4 category, such that $F(\Sigma^i \phi_n) = 0$ for all $i \in \mathbb{Z}$ and all $n \geq 0$, then $F(\underrightarrow{\mathrm{hocolim}}\, x_n) = 0$.*

Proof. With our hypothesis we have $\varinjlim F(x_n) = 0 = \varinjlim^{(1)} F(\Sigma x_n)$, so $F(\underrightarrow{\mathrm{hocolim}}\, x_n) = 0$ by the Milnor exact sequence of Lemma 3.2.1. $\square$

Recall that a pair $(\mathcal{P}, \mathfrak{F})$ consisting of a class of objects $\mathcal{P} \subseteq \mathcal{T}$ and a class of morphisms $\mathfrak{F}$ is called *projective class* if $\Sigma^n(\mathcal{P}) \subseteq \mathcal{P}$ for all $n \in \mathbb{N}$,

$$\mathcal{P} = \{p \in \mathcal{T} \mid \mathcal{T}(p, \phi) = 0 \text{ for all } \phi \in \mathfrak{F}\},$$

$$\mathfrak{F} = \{\phi \in \mathcal{T} \mid \mathcal{T}(p, \phi) = 0 \text{ for all } p \in \mathcal{P}\}$$

and each $x \in \mathcal{T}$ lies in a triangle $\Sigma^{-1}x' \to p \to x \to x'$, with $p \in \mathcal{P}$ and $x \to x'$ in $\mathfrak{F}$ (see [19]). Note that we work only with projective classes which are stable under (de)suspensions; generally it is possible to define a projective class without this condition. Clearly, $\mathcal{P}$ is closed under coproducts and direct factors, $\mathfrak{F}$ is an ideal, and $\mathfrak{F}$ is stable under (de)suspensions. Moreover $\mathcal{P}$ and $\mathfrak{F}$ determine each other. A triangle of the form $x \to y \to z \to \Sigma x$ is called $\mathfrak{F}$*–exact* if the morphism $z \to \Sigma x$ belongs to $\mathfrak{F}$. If this is the case, the morphisms $x \to y$ and $y \to z$ are called $\mathfrak{F}$*–monic*, respectively $\mathfrak{F}$*–epic*.

Let $(\mathcal{P}, \mathfrak{F})$ be a projective class in $\mathcal{T}$. The inclusion functor $\varphi : \mathcal{P} \to \mathcal{T}$ induces a unique right exact functor φ^* making commutative the diagram

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\varphi"] \arrow[d, "H_{\mathcal{P}}"'] & \mathcal{T} \arrow[d, "H_{\mathcal{T}}"] \\ \operatorname{mod}(\mathcal{P}) \arrow[r, "\varphi^*"] & \operatorname{mod}(\mathcal{T}) \end{tikzcd}$$

where $H_{\mathcal{P}}$ and $H_{\mathcal{T}}$ are the respective Yoneda functors. More explicitly,

$$\varphi^*(\mathcal{P}(-, p)) = \mathcal{T}(-, p)$$

for all $p \in \mathcal{P}$, and φ^* is right exact. Moreover since φ is fully–faithful, φ^* has the same property [39, Lemma 2.6].

Fix a projective class $(\mathcal{P}, \mathfrak{F})$ in the triangulated category $\mathcal{T}$. To construct a weak kernel of a morphism $q \to r$ in $\mathcal{P}$ we proceed as follows: The morphism fits into a triangle $x \to q \to r \to \Sigma x$; let $\Sigma^{-1}x' \to p \to x \to x'$ an $\mathfrak{F}$–exact triangle with $p \in \mathcal{P}$; then the composite map $p \to x \to q$ gives the desired weak kernel. Therefore $\operatorname{mod}(\mathcal{P})$ is abelian (for example by [39, Lemma 2.2], but this is also well–known). Moreover the restriction functor

$$\varphi_* : \operatorname{mod}(\mathcal{T}) \to \operatorname{mod}(\mathcal{P}),\ \varphi_*(X) = X \circ \varphi \text{ for all } X \in \operatorname{mod}(\mathcal{T})$$

is well defined and it is the exact right adjoint of φ^*, by [43, Lemma 2].

We know by [19, Lemma 3.2] that a pair $(\mathcal{P}, \mathfrak{F})$ is a projective class, provided that $\mathcal{P}$ is a class of objects closed under direct factors, $\mathfrak{F}$ is an ideal, $\mathcal{P}$ and $\mathfrak{F}$ are *orthogonal* (that means, the composite $p \to x \to x'$ is zero for all $p \in \mathcal{P}$ and all $x \to x'$ in $\mathfrak{F}$) and each object $x \in \mathcal{T}$ lies in an $\mathfrak{F}$–exact triangle $\Sigma^{-1}x' \to p \to x \to x'$, with $p \in \mathcal{P}$. If $\mathcal{S}$ is a set of objects in $\mathcal{T}$, then $\operatorname{Add}(\mathcal{S})$ denotes, as usual, the class of all direct factors of arbitrary coproducts of objects in $\mathcal{S}$. The following lemma is straightforward (see also [19, Definition 5.2 and the following paragraph]):

Lemma 3.2.3. *Consider a Σ-closed set $\mathcal{S}$ of objects in $\mathcal{T}$. Denote by $\mathcal{P} = \operatorname{Add}(\mathcal{S})$, and let $\mathfrak{F}$ be the class of all morphisms ϕ in $\mathcal{T}$ such that $\mathcal{T}(s, \phi) = 0$ for all $s \in \mathcal{S}$. Then $(\mathcal{P}, \mathfrak{F})$ is a projective class.*

We will say that the projective class $(\mathcal{P}, \mathfrak{F})$ given in Lemma 3.2.3 is *induced* by the set $\mathcal{S}$. Note also that if $\mathcal{S}$ is an essentially small subcategory of $\mathcal{T}$, such that $\mathcal{S}$ is Σ=closed, then we will also speak about the projective class induced by $\mathcal{S}$, understanding the projective class induced by a representative set of isomorphism classes of objects in $\mathcal{S}$. If, in particular, κ is a regular cardinal, $\mathcal{S}$ consists of κ–small objects and it is closed under coproducts of less than κ objects (for example if $\mathcal{S}$ is the subcategory of all κ-compact object of $\mathcal{T}$), then $\operatorname{mod}(\mathcal{P})$ is equivalent to the category of all functors $\mathcal{S}^o \to \mathcal{A}b$ which preserve products of less than κ objects, by [42, Lemma 2], category used extensively in [60] as a locally presentable approximation of $\operatorname{mod}(\mathcal{T})$.

Remark 3.2.4. Under the hypotheses of Lemma 3.2.3, a map $x \to y$ in $\mathcal{T}$ is $\mathfrak{F}$–monic ($\mathfrak{F}$–epic) if and only if the induced map $\mathcal{T}(s,x) \to \mathcal{T}(s,y)$ injective (respectively surjective) for all $s \in \mathcal{S}$.

As in [6] and [19], given a projective class $(\mathcal{P},\mathfrak{F})$ in $\mathcal{T}$, we construct two towers of morphisms associated to each $x \in \mathcal{T}$ as follows: We denote $x_0 = \Sigma^{-1}x$. Inductively, if $x_n \in \mathcal{T}$ is given, for $n \in \mathbb{N}$, then there is an $\mathfrak{F}$-exact triangle

$$\Sigma^{-1}x_{n+1} \to p_n \to x_n \overset{\phi_n}{\to} x_{n+1}$$

in $\mathcal{T}$, by definition of a projective class. Consider then the tower:

$$\Sigma^{-1}x = x_0 \overset{\phi_0}{\to} x_1 \overset{\phi_1}{\to} x_2 \overset{\phi_2}{\to} x_3 \to \cdots .$$

Such a tower is called a $\mathfrak{F}$-*phantom tower* of x. The explanation of the terminology is that morphisms ϕ_n belong to $\mathfrak{F}$ for all $n \in \mathbb{N}$, and $\mathfrak{F}$ may be thought as a generalization of the ideal of classical phantom maps in a compactly generated triangulated category. (Clearly, $\mathfrak{F}$ coincides with the ideal of classical phantom maps, provided that the projective class $(\mathcal{P},\mathfrak{F})$ is induced by the full essentially small subcategory consisting of all compact objects.)

Observe that there are more $\mathfrak{F}$-phantom towers associated to the same element $x \in \mathcal{T}$, according with the choices of the $\mathfrak{F}$-epic map $p_n \to x_n$ at each step $n \in \mathbb{N}$. The analogy with projective resolutions in abelian categories is obvious.

Choose an $\mathfrak{F}$-phantom tower of $x \in \mathcal{T}$ as in the definition above. We denote by ϕ^n the composed map $\phi_{n-1}\dots\phi_1\phi_0 : \Sigma^{-1}x \to x_n$, for all $n \in \mathbb{N}^*$, and we set $\phi^0 = 1_{\Sigma^{-1}x}$. Then let x^n be defined, uniquely up to a non unique isomorphism, by the triangle $\Sigma^{-1}x \overset{\phi^n}{\to} x_n \to x^n \to x$. The octahedral axiom allows us to complete the commutative diagram

$$\begin{tikzcd}
 & p_n \arrow[r, equal] \arrow[d] & p_n \arrow[d] & \\
\Sigma^{-1}x \arrow[r, "\phi^n"] \arrow[d, equal] & x_n \arrow[r] \arrow[d, "\phi_n"] & x^n \arrow[r] \arrow[d] & x \arrow[d, equal] \\
\Sigma^{-1}x \arrow[r, "\phi^{n+1}"] & x_{n+1} \arrow[r] \arrow[d] & x^{n+1} \arrow[r] \arrow[d] & x \\
 & \Sigma p_n \arrow[r, equal] & \Sigma p_n &
\end{tikzcd}$$

with the triangle in the second column.

Therefore we obtain an another tower of objects

$$0 = x^0 \to x^1 \to x^2 \to x^3 \to \cdots ,$$

where for each $n \in \mathbb{N}$ we have a triangle $p_n \to x^n \to x^{n+1} \to \Sigma p_n$, with $p_n \in \mathcal{P}$ chosen in the construction of the above $\mathfrak{F}$-phantom tower. Such a tower is called a $\mathfrak{F}$-*cellular tower* of $x \in \mathcal{T}$.

Considering homotopy colimits of the $\mathfrak{F}$-phantom and $\mathfrak{F}$-cellular towers, we obtain a sequence

$$\Sigma^{-1}x \to \underrightarrow{\mathrm{hocolim}}\, x_n \to \underrightarrow{\mathrm{hocolim}}\, x^n \to x.$$

It is not known whether the induced sequence can be chosen to be a triangle (see [6, p. 302]). However the answer to this question is yes, provided that $\mathcal{T}$ is the homotopy category of a suitable stable closed model category in the sense of [31], or $\mathcal{T}$ is the underlying category of a stable derivator (see [38, Corollary 11.4]).

Proposition 3.2.5. *Let $(\mathcal{P}, \mathfrak{F})$ be a projective class in $\mathcal{T}$, and let denote by $\varphi : \mathcal{P} \to \mathcal{T}$ the inclusion functor. For every $x \in \mathcal{T}$ we consider an $\mathfrak{F}$-phantom tower and an $\mathfrak{F}$-cellular tower as above. Then we have an exact sequence*

$$0 \to \coprod(\varphi_* \circ H_{\mathcal{T}})(x^n) \overset{1-shift}{\longrightarrow} \coprod(\varphi_* \circ H_{\mathcal{T}})(x^n) \to (\varphi_* \circ H_{\mathcal{T}})(x) \to 0,$$

where $\varphi_* : \mathrm{mod}(\mathcal{T}) \to \mathrm{mod}(\mathcal{P})$ is the restriction functor. Consequently

$$\underrightarrow{\mathrm{colim}}(\varphi_* \circ H_{\mathcal{T}})(x^n) \cong (\varphi_* \circ H_{\mathcal{T}})(x) \text{ and } \underrightarrow{\mathrm{colim}}^{(1)}(\varphi_* \circ H_{\mathcal{T}})(x^n) = 0.$$

Proof. By applying the functor $\varphi_* \circ H_{\mathcal{T}}$ to the diagram above defining an $\mathfrak{F}$-cellular tower associated to x, we obtain a commutative diagram in the abelian category with coproducts $\mathrm{mod}(\mathcal{P})$:

$$\begin{tikzcd}
0 \arrow[r] & (\varphi_* \circ H_{\mathcal{T}})(x_n) \arrow[r] \arrow[d, "0"'] & (\varphi_* \circ H_{\mathcal{T}})(x^n) \arrow[r] \arrow[d] & (\varphi_* \circ H_{\mathcal{T}})(x) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\varphi_* \circ H_{\mathcal{T}})(x_{n+1}) \arrow[r] & (\varphi_* \circ H_{\mathcal{T}})(x^{n+1}) \arrow[r] & (\varphi_* \circ H_{\mathcal{T}})(x) \arrow[r] & 0
\end{tikzcd}$$

The conclusion follows by [44, Lemma 7.1.2]. □

Consider a regular cardinal κ. Recall that by κ-coproducts we understand coproducts of less that κ-objects.

Proposition 3.2.6. *Let κ be a regular cardinal and let $(\mathcal{P}, \mathfrak{F})$ be a projective class in $\mathcal{T}$. Denote by $\varphi : \mathcal{P} \to \mathcal{T}$ the inclusion functor. Then the functor $\varphi_* : \mathrm{mod}(\mathcal{T}) \to \mathrm{mod}(\mathcal{P})$, $\varphi_*(X) = X \circ \varphi$ preserves κ–coproducts if and only if $\mathfrak{F}$ is closed under κ–coproducts (of maps).*

Proof. The exact functor φ_* having a fully–faithful left adjoint induces an equivalence $\mathrm{mod}(\mathcal{T})/\operatorname{Ker}\varphi_* \to \mathrm{mod}(\mathcal{P})$. Since $\mathrm{mod}(\mathcal{T})$ is AB4, we know that φ_* preserves κ–coproducts if and only if $\operatorname{Ker}\varphi_*$ is closed under κ–coproducts. Obviously $\mathfrak{F} = \{\phi \mid (\varphi_* \circ H_{\mathcal{T}})(\phi) = 0\}$. Using the proof of [40, Section 3], we observe that

$$\operatorname{Ker}\varphi_* = \{X \in \mathrm{mod}(\mathcal{T}) \mid X \cong \operatorname{im} H_{\mathcal{T}}(\phi) \text{ for some } \phi \in \mathfrak{F}\}.$$

Now suppose $\mathfrak{F}$ to be closed under κ–coproducts, and let $\{M_\lambda \mid \lambda \in \Lambda\}$ be a set of objects in $\operatorname{Ker}\varphi_*$, with the cardinality less than κ. Thus $M_\lambda \cong \operatorname{im} H_{\mathcal{T}}(\phi_\lambda)$ for some $\phi_\lambda \in \mathfrak{F}$, for all $\lambda \in \Lambda$. Therefore, using again condition AB4 (coproducts in $\mathrm{mod}(\mathcal{T})$ are exact, so they commute with images), we obtain:

$$\coprod_{\lambda\in\Lambda} M_\lambda \cong \coprod_{\lambda\in\Lambda} \operatorname{im} H_{\mathcal{T}}(\phi_\lambda) \cong \operatorname{im}\left(\coprod_{\lambda\in\Lambda} H_{\mathcal{T}}(\phi_\lambda)\right) \cong \operatorname{im} H_{\mathcal{T}}\left(\coprod_{\lambda\in\Lambda} \phi_\lambda\right),$$

showing that $\coprod_{\lambda\in\Lambda} M_\lambda \in \operatorname{Ker}\varphi_*$.

Conversely, if $\operatorname{Ker}\varphi_*$ is closed under κ–coproducts, and $\{\phi_\lambda \mid \lambda \in \Lambda\}$ is a set of maps in $\mathfrak{F}$, with the cardinality less than κ, then

$$\varphi_*\left(\operatorname{im} H_{\mathcal{T}}\left(\coprod_{\lambda\in\Lambda} \phi_\lambda\right)\right) = \varphi_*\left(\coprod_{\lambda\in\Lambda} \operatorname{im} H_{\mathcal{T}}(\phi_\lambda)\right) = 0,$$

so $\mathfrak{F}$ is closed under κ–coproducts. $\square$

We call *κ–perfect* the projective class $(\mathcal{P}, \mathfrak{F})$ if the equivalent conditions of Proposition 3.2.6 hold true. The projective class will be called *perfect* if it is κ–perfect for all regular cardinals κ, that is, $\mathfrak{F}$ is closed under arbitrary coproducts. Following [19], we say that a projective class $(\mathcal{P}, \mathfrak{F})$ *generates* $\mathcal{T}$ if for any $x \in \mathcal{T}$, we have $x = 0$ provided that $\mathcal{T}(p, x) = 0$, for each $p \in \mathcal{P}$. Immediately, we can see that $(\mathcal{P}, \mathfrak{F})$ generates $\mathcal{T}$ if and only if $\varphi \circ H_{\mathcal{T}} : \mathcal{T} \to \mathrm{mod}(\mathcal{P})$ reflects isomorphisms, that is, if $\alpha : x \to y$ is a morphism in $\mathcal{T}$ such that the induced morphism $(\varphi\circ H_{\mathcal{T}})(\alpha)$ is an isomorphism in $\mathrm{mod}(\mathcal{P})$, then α is an isomorphism in $\mathcal{T}$, where $\varphi : \mathcal{P} \to \mathcal{T}$ denotes, as usual, the inclusion functor. Another equivalent statement is $\mathfrak{F}$ does not contain non–zero identity maps. Consider now an essentially small subcategory $\mathcal{S}$ of $\mathcal{T}$ which is closed under suspensions and desuspensions, and $(\mathcal{P}, \mathfrak{F})$ the projective class induced by $\mathcal{S}$. Since coproducts of triangles are triangles, we conclude by Remark 3.2.4 that $\mathfrak{F}$ is closed under coproducts exactly if $\mathcal{S}$ satisfies the following condition: If $x_i \to y_i$ with $i \in I$ is a family of maps, such that $\mathcal{T}(s, x_i) \to \mathcal{T}(s, y_i)$ is surjective for all $i \in I$, then the induced map $\mathcal{T}(s, \coprod x_i) \to \mathcal{T}(s, \coprod y_i)$ is also surjective. Thus $(\mathcal{P}, \mathfrak{F})$ perfectly generates $\mathcal{T}$ in the sense above if and only if $\mathcal{S}$ perfectly generates $\mathcal{T}$ in the sense given in [44, Section 5] (see also [43] for a version relativized at the cardinal $\kappa = \aleph_1$).

Lemma 3.2.7. *Consider a tower $x_0 \stackrel{\phi_0}{\to} x_1 \stackrel{\phi_1}{\to} x_2 \stackrel{\phi_2}{\to} x_3 \to \cdots$ in $\mathcal{T}$. If $(\mathcal{P}, \mathfrak{F})$ is an $\aleph_1$–perfect projective class in $\mathcal{T}$ and $\phi_n \in \mathfrak{F}$ for all $n \geq 0$, then*

$$\underrightarrow{\operatorname{hocolim}}\, x_n = 0.$$

Proof. We apply Corollary 3.2.2 to the homological functor, which preserves countable coproducts $\varphi_* \circ H_T : \mathcal{T} \to \mathrm{mod}(\mathcal{P})$, where $\varphi : \mathcal{P} \to \mathcal{T}$ is the inclusion functor. □

Proposition 3.2.8. *If $(\mathcal{P}, \mathfrak{F})$ is a $\aleph_1$–perfect projective class in $\mathcal{T}$, then a necessary and sufficient condition for $(\mathcal{P}, \mathfrak{F})$ to generate $\mathcal{T}$ is*

$$\lim_{n\in\mathbb{N}} \mathcal{T}(x_n, y) = 0 = \lim_{n\in\mathbb{N}}{}^{(1)} \mathcal{T}(x_n, y),$$

for all $x, y \in \mathcal{T}$ and any choice

$$x = x_0 \xrightarrow{\phi_0} x_1 \xrightarrow{\phi_1} x_2 \xrightarrow{\phi_2} x_3 \to \cdots ,$$

of an $\mathfrak{F}$-phantom tower of x. Here by $\lim^{(1)}$ we understand the first derived functor of the limit.

Proof. Let show the sufficiency first. If $x \in \mathcal{T}$ has the property $\mathcal{T}(p, x) = 0$ for all $p \in \mathcal{P}$, then $1_x \in \mathfrak{F}$ and a $\mathfrak{F}$-phantom tower of x is

$$x = x_0 \xrightarrow{1_x} x_1 = x \xrightarrow{1_x} x_2 = x \to \cdots .$$

Then $0 = \lim_{n\in\mathbb{N}} \mathcal{T}(x_n, x) = \mathcal{T}(x, x)$, so $x = 0$.

Now we show the necessity. Let $x, y \in \mathcal{T}$ and consider an $\mathfrak{F}$-phantom tower of x as above. Applying the functor $\mathcal{T}(-, y)$ to this tower, we obtain a sequence of abelian groups:

$$\mathcal{T}(x, y) = \mathcal{T}(x_0, y) \xleftarrow{(\phi_0)_*} \mathcal{T}(x_1, y) \xleftarrow{(\phi_1)_*} \mathcal{T}(x_2, y) \xleftarrow{(\phi_2)_*} \mathcal{T}(x_3, y) \leftarrow \cdots .$$

Computing the derived functors of the limit of such a sequence in the usual manner, we know that $\lim^{(n)}$ is zero for $n \geq 2$ and lim, $\lim^{(1)}$ are given by the exact sequence:

$$0 \to \lim_{n\in\mathbb{N}} \mathcal{T}(x_n, y) \to \prod_{n\in\mathbb{N}} \mathcal{T}(x_n, y) \xrightarrow{(1-\phi)_*} \prod_{n\in\mathbb{N}} \mathcal{T}(x_n, y) \to \lim_{n\in\mathbb{N}}{}^{(1)} \mathcal{T}(x_n, y) \to 0,$$

where $\phi : \coprod_{n\in\mathbb{N}} x_n \to \coprod_{n\in\mathbb{N}} x_n$ is constructed as in the Definition of the homotopy colimit 1.2.6. Applying $\mathcal{T}(p, -)$ to the commutative squares which define ϕ, we obtain also commutative squares:

$$\begin{tikzcd}
\mathcal{T}(p, x_n) \arrow[r] \arrow[d, "0=\mathcal{T}(p,\phi_n)"'] & \mathcal{T}(p, \coprod_{n\in\mathbb{N}} x_n) \arrow[d, "\mathcal{T}(p,\phi)"] \\
\mathcal{T}(p, x_{n+1}) \arrow[r] & \mathcal{T}(p, \coprod_{n\in\mathbb{N}} x_n)
\end{tikzcd} \quad (n \in \mathbb{N}),$$

for all $p \in \mathcal{P}$. According to Proposition 3.2.6, the $\aleph_1$-perfectness of $(\mathcal{P}, \mathfrak{F})$ means that $\mathcal{T}(-, \coprod_{n\in\mathbb{N}} x_n)|_{\mathcal{P}}$ is the coproduct in $\mathrm{mod}(\mathcal{P})$ of the set

$$\{\mathcal{T}(-, x_n)|_{\mathcal{P}} \mid n \in \mathbb{N}\},$$

thus we deduce $\mathcal{T}(p, \phi) = 0$. Now $\mathcal{T}(p, 1 - \phi) = \mathcal{T}(p, 1) - \mathcal{T}(p, \phi) = \mathcal{T}(p, 1)$ is an isomorphism, for all $p \in \mathcal{P}$, so $1 - \phi$ is an isomorphism, because $(\mathcal{P}, \mathfrak{F})$ generates $\mathcal{T}$. Consequently

$$\lim_{n\in\mathbb{N}} \mathcal{T}(x_n, y) = 0 = \lim_{n\in\mathbb{N}}{}^{(1)} \mathcal{T}(x_n, y).$$

$\square$

Remark 3.2.9. The hypotheses of Proposition 3.2.8 are almost identical with those of [19, Proposition 4.4], except the fact that we require, in addition, the $\aleph_1$-perfectness for $(\mathcal{P}, \mathfrak{F})$. Moreover, the conclusion of [19, Proposition 4.4] (namely: the Adams spectral sequence abutting $\mathcal{T}(x, y)$ is conditionally convergent) is equivalent to our conclusion (lim and $\lim^{(1)}$ to be zero). The proofs are also almost identical. Despite that, we have given a detailed proof, because, without our additional condition, we do not see how we can conclude, with our notations, that $\mathcal{T}(p, \phi) = 0$. Thus we fill a gap existing in the proof of [19, Proposition 4.4], due to the missing assumption of $\aleph_1$-perfectness. On the other hand, we do not have a counterexample showing that the conclusion cannot be inferred without this assumption, so the problem is open. Note also that the terms of the Adams spectral sequence of [19] do not depend, for sufficiently large indices, of the choice of the $\mathfrak{F}$-projective resolution of $x \in \mathcal{T}$, so the conclusion of Proposition 3.2.8 may be formulated simply: The Adams spectral sequence abutting $\mathcal{T}(x, y)$ is conditionally convergent, for any two $x, y \in \mathcal{T}$.

Theorem 3.2.10. *Let $(\mathcal{P}, \mathfrak{F})$ be an $\aleph_1$–perfectly generating projective class in $\mathcal{T}$. Then for every $x \in \mathcal{T}$, and every choice*

$$0 = x^0 \to x^1 \to x^2 \to x^3 \to \cdots$$

of an $\mathfrak{F}$-cellular tower for x we have $\underrightarrow{\mathrm{hocolim}}\, x^n \cong x$.

Proof. The homotopy colimit of the $\mathfrak{F}$-cellular tower above is constructed via triangle

$$\coprod_{n\in\mathbb{N}} x^n \xrightarrow{1-shift} \coprod_{n\in\mathbb{N}} x^n \to \underrightarrow{\mathrm{hocolim}}\, x^n \to \Sigma \coprod_{n\in\mathbb{N}} x^n.$$

We apply to this triangle the homological functor $\varphi_* \circ H_{\mathcal{T}}$ which commutes with countable coproducts. Comparing the resulting exact sequence with the exact sequence given by Proposition 3.2.5, we obtain a unique isomorphism

$$(\varphi_* \circ H_{\mathcal{T}})(\underrightarrow{\mathrm{hocolim}}\, x^n) \to (\varphi_* \circ H_{\mathcal{T}})(x),$$

which must be induced by the map $\underrightarrow{\mathrm{hocolim}}\, x^n \to x$. The hypothesis $(\mathcal{P}, \mathfrak{F})$ generates $\mathcal{T}$ tells us that $\underrightarrow{\mathrm{hocolim}}\, x^n \cong x$. $\square$

Recall that an $\aleph_1$–localizing subcategory of $\mathcal{T}$ means a triangulated subcategory closed under countable coproducts.

Corollary 3.2.11. *If $(\mathcal{P},\mathfrak{F})$ is an $\aleph_1$–perfectly generating projective class in $\mathcal{T}$, then $\mathcal{T}$ is the smallest $\aleph_1$–localizing subcategory of $\mathcal{T}$, which contains $\mathcal{P}$.*

Proof. Let $x \in \mathcal{T}$ and let

$$0 = x^0 \to x^1 \to x^2 \to x^3 \to \cdots$$

be an $\mathfrak{F}$-cellular tower for x. Since for every $n \geq 0$ there exits a triangle $p_n \to x_n \to x_{n+1} \to \Sigma p_n$, with $p_n \in \mathcal{P}$ (see the definition of an $\mathfrak{F}$-cellular tower), we may see inductively that x_n belongs to the smallest triangulated subcategory of $\mathcal{T}$ which contains $\mathcal{P}$. Now $\underrightarrow{\mathrm{hocolim}}\, x^n$ belongs to the smallest $\aleph_1$-localizing subcategory of $\mathcal{T}$ which contains $\mathcal{P}$, and the conclusion follows by Theorem 3.2.10. □

Remark 3.2.12. Let $(\mathcal{P},\mathfrak{F})$ be an $\aleph_1$–perfectly generating projective class in $\mathcal{T}$, and $x \in \mathcal{T}$. If we chose an $\mathfrak{F}$-phantom tower

$$\Sigma^{-1}x = x_0 \xrightarrow{\phi_0} x_1 \xrightarrow{\phi_1} x_2 \xrightarrow{\phi_2} x_3 \to \cdots$$

and an $\mathfrak{F}$-cellular tower

$$0 = x^0 \to x^1 \to x^2 \to x^3 \to \cdots$$

for x, then $\underrightarrow{\mathrm{hocolim}}\, x_n = 0$ by Lemma 3.2.7, and $\underrightarrow{\mathrm{hocolim}}\, x^n \cong x$ by Theorem 3.2.10. Thus the triangle $\Sigma^{-1}x \to \underrightarrow{\mathrm{hocolim}}\, x_n \to \underrightarrow{\mathrm{hocolim}}\, x^n \to x$ is trivially exact.

Remark 3.2.13. A filtration analogous to that of Theorem 3.2.10, for the case of well–generated triangulated categories may be found in [60, Lemma B 1.3].

For two projective classes $(\mathcal{P},\mathfrak{F})$ and $(\mathcal{Q},\mathfrak{G})$, we define the *product* by

$$\begin{aligned}\mathcal{P} * \mathcal{Q} = \mathrm{add}\{x \in \mathcal{T} \mid &\text{there is a triangle} \\ &q \to x \to p \to \Sigma q \text{ with } p \in \mathcal{P}, q \in \mathcal{Q}\},\end{aligned}$$

and $\mathfrak{F} * \mathfrak{G} = \{\phi\psi \mid \phi \in \mathfrak{F}, \psi \in \mathfrak{G}\}$. Generally by add we understand the closure under finite coproducts and direct factors. Since in our case the closure under arbitrary coproducts is automatically fulfilled, add means here simply the closure under direct factors. Thus $(\mathcal{P} * \mathcal{Q}, \mathfrak{F} * \mathfrak{G})$ is a projective class, by [19, Proposition 3.3].

If $(\mathcal{P}_i,\mathfrak{F}_i)$ for $i \in I$ is a family of projective classes, then

$$\left(\mathrm{Add}\left(\bigcup_I \mathcal{P}_i\right), \bigcap_I \mathfrak{F}_i\right)$$

is also a projective class by [19, Proposition 3.1], called the *meet* of the above family.

In a straightforward manner we may use the octahedral axiom in order to show that the product defined above is associative. We may also observe without difficulties that the product of two (respectively the meet of a family of) κ-perfect projective classes is also κ-perfect, where κ is an arbitrary regular cardinal.

Consider now a projective class $(\mathcal{P}, \mathfrak{F})$ in $\mathcal{T}$. We define inductively $\mathcal{P}^{*0} = \{0\}$, $\mathfrak{F}^{*0} = \mathcal{T}^{\to}$ and $\mathcal{P}^{*i} = \mathcal{P} * \mathcal{P}^{*(i-1)}$, $\mathfrak{F}^{*i} = \mathfrak{F} * \mathfrak{F}^{*(i-1)}$, for every non–limit ordinal $i > 0$. If i is a limit ordinal then $(\mathcal{P}^{*i}, \mathfrak{F}^{*i})$ is defined as the meet of all $(\mathcal{P}^{*j}, \mathfrak{F}^{*j})$ with $j < i$. Therefore $(\mathcal{P}^{*i}, \mathfrak{F}^{*i})$ is a projective class for every ordinal i, which is called the *i-th power* of the projective class of $(\mathcal{P}, \mathfrak{F})$ (see also [19], for the case of ordinals less or equal to the first infinite ordinal). Clearly we have $\mathcal{P}^{*j} \subseteq \mathcal{P}^{*i}$, for all ordinals $j \leq i$.

Remark 3.2.14. We can inductively see that for $x \in \mathcal{T}$ it holds $x^n \in \mathcal{P}^{*n}$ for all $n \in \mathbb{N}$, where x^n is the n–th term of an $\mathfrak{F}$-cellular tower of x.

For example, if $\mathcal{T}$ is compactly generated, and $\mathcal{T}^c$ denotes the subcategory of all compact objects, then the projective class induced by $\mathcal{T}^c$ is obviously perfect, thus we obtain immediate consequence of Theorem 3.2.10:

Corollary 3.2.15. *[6, Corollary 6.9] If $\mathcal{T}$ is compactly generated then any object $x \in \mathcal{T}$ is the homotopy colimit $\underrightarrow{\mathrm{hocolim}}\, x^n$ of a tower $x^0 \to x^1 \to \cdots$, where $x^n \in \mathrm{Add}(\mathcal{T}^c)^{*n}$, for all $n \in \mathbb{N}$.*

Consider a contravariant functor $F : \mathcal{T} \to \mathcal{A}b$. For a full subcategory $\mathcal{C}$ of $\mathcal{T}$, we consider the comma category $\mathcal{C}/F$ with the objects being pairs of the form (x, a), where $x \in \mathcal{C}$ and $a \in F(x)$, and maps

$$(\mathcal{C}/F)((x, a), (y, b)) = \{\alpha \in \mathcal{T}(x, y) \mid F(\alpha)(b) = a\}.$$

Motivated by [61] it is interesting to find weak terminal objects in $\mathcal{T}/F$, that is objects $(t, b) \in \mathcal{T}/F$, such that for every $(x, a) \in \mathcal{T}/F$ there is a map $(x, a) \to (t, b) \in (\mathcal{T}/F)^{\to}$. Another equivalent formulation of this fact is that the natural transformation $\mathcal{T}(-, t) \to F$ which corresponds under the Yoneda isomorphism to $b \in F(t)$ is an epimorphism. The statement a) of the following lemma is proved by the same argument as [61, Lemma 2.3]. We include a sketch of the proof for the convenience of the reader.

Lemma 3.2.16. *Let $F : \mathcal{T} \to \mathcal{A}b$ be a cohomological functor which sends coproducts into products.*

a) If $(\mathcal{P}, \mathfrak{F})$ and $(\mathcal{Q}, \mathfrak{G})$ are projective classes in $\mathcal{T}$ such that $(\mathcal{P}, \mathfrak{F})$ is induced by a set and $\mathcal{Q}/F$ has a weak terminal object, then $(\mathcal{P} * \mathcal{Q})/F$ has a weak terminal object.

b) If $(\mathcal{P}_i, \mathfrak{F}_i)$, $i \in I$ are projective classes in $\mathcal{T}$ with the meet $(\mathcal{P}, \mathfrak{F})$, and $\mathcal{P}_i/F$ has a weak terminal object for all $i \in I$ then $\mathcal{P}/F$ has a weak terminal object.

Proof. a) Let (q,d) be a weak terminal object in $\mathcal{Q}/F$, and let $\mathcal{S}$ be a set which induces the projective class $(\mathcal{P},\mathfrak{F})$. Obviously $\mathcal{P}/F$ has a weak terminal object (p,c). Consider an object $(y,a)\in(\mathcal{P}*\mathcal{Q})/F$. Thus there is a triangle $x\to y\to z\to\Sigma x$, with $x\in\mathcal{Q}$ and $z\in\mathcal{P}$. We have $z\amalg z'=\coprod_{i\in I}s_i$ for some $z'\in\mathcal{T}$. We construct the commutative diagram in $\mathcal{T}$ whose rows are triangles:

$$\begin{tikzcd}
x \arrow[r] \arrow[d, equal] & y \arrow[r] \arrow[d] & z \arrow[r] \arrow[d] & \Sigma x \arrow[d, equal] \\
x \arrow[r, "\alpha"] \arrow[d, "f"'] & y\amalg z' \arrow[r] \arrow[d, "g"] & \coprod_{i\in I}s_i \arrow[r] \arrow[d, equal] & \Sigma x \arrow[d] \\
q \arrow[r, "\beta"'] & y_1 \arrow[r] & \coprod_{i\in I}s_i \arrow[r, "\gamma"'] & \Sigma q
\end{tikzcd}$$

We proceeded as follows: The triangle on the second row is obtained as the coproduct of the initial one with $0\to z'\to z'\to 0$, and the maps are the canonical injections. For $d'=F(\alpha)(a,0)\in F(x)$, there is a map $f:(x,d')\to(q,d)\in(\mathcal{Q}/F)^{\to}$, since (q,d) is weak terminal. The first bottom square of the diagram above is homotopy push–out (see [60, Definition 1.4.1 and Lemma 1.4.4]). Clearly $y_1\in\mathcal{P}*\mathcal{Q}$. Since F is cohomological, there is $a_1\in F(y_1)$ such that $F(\beta)(a_1)=d$ and $F(g)(a_1)=(a,0)$. So if we find a map $(y_1,a_1)\to(t,b)\in((\mathcal{P}*\mathcal{Q})/F)^{\to}$ for a fixed object (t,b), then the conclusion follows.

If we denote by $J\subseteq\bigcup_{s\in\mathcal{S}}\mathcal{T}(s,\Sigma q)$ the set of all maps $s_i\to\coprod_{i\in I}s_i\to\Sigma q$, then γ factors as $\coprod_{i\in I}s_i\xrightarrow{\nabla}\coprod_{s\in J}s\to\Sigma q$, where ∇ is a split epimorphism. Hence the fibre of γ is isomorphic to $y_J\amalg z''$, for some $z''\in\mathcal{P}$ and y_J defined as the fibre of the canonical map $\coprod_{s\in J}s\to\Sigma q$. Therefore (y,a) maps to $(t,b)=(t'\amalg p,(b',c))$ where

$$(t',b')=\left(\coprod_{J\subseteq\bigcup_{s\in\mathcal{S}}\mathcal{T}(s,\Sigma q)}\left(\coprod_{u\in F(y_J)}(y_J,u)\right)\right),$$

so the object (t,b) is weak terminal in $(\mathcal{P}*\mathcal{Q})/F$.

b) If $(t_i,a_i)\in\mathcal{P}_i/F$ is a weak terminal object, then $(\coprod_{i\in I}t_i,(a_i)_{i\in I})$ is a weak terminal object in $\mathcal{P}/F$. □

By transfinite induction we obtain:

Lemma 3.2.17. *Let $(\mathcal{P},\mathfrak{F})$ be a projective class in $\mathcal{T}$ which is induced by a set. For every ordinal i and every cohomological functor $F:\mathcal{T}\to\mathcal{A}b$ which sends coproducts into products, the category $\mathcal{P}^{*i}/F$ has a weak terminal object.*

Remark 3.2.18. For finite ordinals, Lemma 3.2.17 is the same as [61, Lemma 2.3]. Note also that Neeman defined the operation $*$ without to assume the closure under direct summands, but for a subcategory $\mathcal{C}$ of $\mathcal{T}$ such that (t,b) is weak terminal in $\mathcal{C}/F$, the same object is weak terminal in $\operatorname{add}\mathcal{C}/F$ too.

3.3 Well–generation and deconstructibility

The following Proposition is a generalization of Lemma 3.1.2. Despite the fact the idea is the same we sketch here this argument (in the dual form appropriate to the present approach), because the hypotheses are dramatically modified.

Proposition 3.3.1. *Let $(\mathcal{P}, \mathfrak{F})$ be an $\aleph_1$–perfectly generating projective class in $\mathcal{T}$, and let $F : \mathcal{T} \to \mathcal{A}b$ be a cohomological functor which sends coproducts into products. Suppose also that every category $\mathcal{P}^{*n}/F$ has a weak terminal object (t^n, b_n), for $n \in \mathbb{N}$. Then $\mathcal{T}/F$ has a weak terminal object.*

Proof. Denote by I the set of all towers $0 = t^0 \xrightarrow{\tau_0} t^1 \xrightarrow{\tau_1} t^2 \to \cdots$, satisfying $F(\tau_n)(b_{n+1}) = b_n$, for all $n \in \mathbb{N}$. The set I is not empty since for all $n \in \mathbb{N}$, we have $t^n \in \mathcal{P}^{*n} \subseteq \mathcal{P}^{*(n+1)}$ and (t^{n+1}, b_{n+1}) is weak terminal in $\mathcal{P}^{*(n+1)}/F$. Denote also by t_i the homotopy colimits of the tower $i \in I$, and chose $b_i \in F(t_i)$ an element which maps into $(b_n)_{n\in\mathbb{N}}$ via the surjective (see the dual of Lemma 3.2.1) map $F(t_i) \to \lim\limits_{n\in\mathbb{N}} F(t^n)$. We claim that

$$(t, b) = \left(\coprod_{i\in I} t_i, (b_i)_{i\in I} \right) \in \mathcal{T}/F$$

is a weak terminal object.

In order to prove our claim, let $x \in \mathcal{T}$. As we have seen in Theorem 3.2.10, it is isomorphic to the the homotopy colimit of its $\mathfrak{F}$-cellular tower $0 = x^0 \xrightarrow{\alpha_0} x^1 \xrightarrow{\alpha_1} x^2 \to \cdots$, associated with a choice of an $\mathfrak{F}$-phantom tower. Thus consider the commutative diagram, whose rows are exact by Lemma 3.2.1 and whose vertical arrows are induced by the natural transformation corresponding to $b \in F(t)$ via the Yoneda isomorphism:

$$\begin{tikzcd}
0 \arrow[r] & \lim^{(1)} \mathcal{T}(\Sigma x^n, t) \arrow[r] \arrow[d] & \mathcal{T}(x, t) \arrow[r] \arrow[d] & \lim \mathcal{T}(x^n, t) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \lim^{(1)} F(\Sigma x^n) \arrow[r] & F(x) \arrow[r] & \lim F(x^n) \arrow[r] & 0
\end{tikzcd}$$

If we would prove that the two extreme vertical arrows are surjective, then the middle arrow enjoys the same property and our work would be done.

For $n \in \mathbb{N}$, we know that $\Sigma x^n \in \mathcal{P}^{*n}$ and (t^n, b_n) is weak terminal in $\mathcal{P}^{*n}$, so there is a map $(\Sigma x^n, a_n) \to (t^n, b_n) \in (\mathcal{P}^{*n}/F)^{\to}$ for every element $a_n \in F(\Sigma x^n)$. Because $I \neq \emptyset$, there exists $i \in I$, hence we obtain a map

$$(\Sigma x^n, a_n) \to (t^n, b_n) \to (t_i, b_i) \to (t, b) \in (\mathcal{T}/F)^{\to}$$

showing that the natural map $\mathcal{T}(\Sigma x^n, t) \to F(\Sigma x^n)$ is surjective. Therefore the first vertical map in the commutative diagram above is surjective as we may see from the following commutative diagram with exact rows:

$$\begin{tikzcd}
\prod \mathcal{T}(\Sigma x^n, t) \arrow[r, "1-shift"] \arrow[d] & \prod \mathcal{T}(\Sigma x^n, t) \arrow[r] \arrow[d] & \lim^{(1)} \mathcal{T}(\Sigma x^n, t) \arrow[r] \arrow[d] & 0 \\
\prod F(\Sigma x^n) \arrow[r, "1-shift"] & \prod F(\Sigma x^n) \arrow[r] & \lim^{(1)} F(\Sigma x^n) \arrow[r] & 0
\end{tikzcd}$$

Let show now that the map $\lim \mathcal{T}(x^n, t) \to \lim F(x^n)$ is surjective too. Consider an element $(a_n) \in \lim F(x^n)$, that is $a_n \in F(x^n)$ such that $a_n = F(\alpha_n)(a_{n+1})$ for all $n \in \mathbb{N}$. We want to construct a commutative diagram

$$\begin{tikzcd}
x^0 \arrow[r, "\alpha_0"] \arrow[d, "f^0"] & x^1 \arrow[r, "\alpha_1"] \arrow[d, "f^1"] & x^2 \arrow[r] \arrow[d, "f^2"] & \cdots \\
t^0 \arrow[r, "\tau_0"'] & t^1 \arrow[r, "\tau_1"'] & t^2 \arrow[r] & \cdots
\end{tikzcd}$$

such that the bottom row is a tower in I and $F(f^n)(b^n) = a_n$ for all $n \in \mathbb{N}$. We proceed inductively as follows: $f^0 = 0$ and f^1 comes from the fact that (t^1, b_1) is weak terminal in $\mathcal{P}/F$. Suppose that the construction is done for the first n steps. Further we construct a commutative diagram in $\mathcal{T}$, where the rows are triangles and the second square is homotopy push–out (see [60, Definition 1.4.1 and Lemma 1.4.4]):

$$\begin{tikzcd}
p_n \arrow[r] \arrow[d, equal] & x^n \arrow[r, "\alpha_n"] \arrow[d, "f^n"'] & x^{n+1} \arrow[r] \arrow[d] & \Sigma p_n \arrow[d, equal] \\
p_n \arrow[r] & t^n \arrow[r] & y^{n+1} \arrow[r] & \Sigma p_n
\end{tikzcd}$$

By construction $p_n \in \mathcal{P}$, hence $y^{n+1} \in \mathcal{P}^{*(n+1)}$. On the other hand y^{n+1} is obtained via the triangle

$$x^n \overset{\begin{pmatrix} \alpha_n \\ -f^n \end{pmatrix}}{\longrightarrow} x^{n+1} \amalg t^n \to y^{n+1} \to \Sigma x^n,$$

therefore the sequence

$$F(y^{n+1}) \to F(x^{n+1}) \times F(t^n) \overset{(F(\alpha_n), -F(f^n))}{\longrightarrow} F(x^n)$$

is exact in $\mathcal{A}b$. Because $F(\alpha_n)(a_{n+1}) - F(f^n)(b_n) = a_n - a_n = 0$, we obtain an element $b'_{n+1} \in F(y^{n+1})$ which is sent to (a_{n+1}, b_n) by the first map in the exact sequence above. Thus the two maps constructed in the homotopy push-out square above are actually maps $(x^{n+1}, a_{n+1}) \to (y^{n+1}, b'_{n+1})$ respectively $(t^n, b_n) \to (y^{n+1}, b'_{n+1})$ in $\mathcal{P}^{*(n+1)}/F$. Since (t^{n+1}, b_{n+1}) is weak terminal in

$\mathcal{P}^{*(n+1)}/F$, they can be composed with a map $(y^{n+1}, b'_{n+1}) \to (t^{n+1}, b_{n+1}) \in (\mathcal{P}^{*(n+1)}/F)^{\to}$, in order to obtain a commutative square

$$\begin{tikzcd} x^n \arrow[r, "\alpha_n"] \arrow[d, "f^n"'] & x^{n+1} \arrow[d, "f^{n+1}"] \\ t^n \arrow[r, "\tau_n"'] & t^{n+1} \end{tikzcd}$$

as desired. Denote by $i \in I$ the tower constructed above. We have a composed map $F(t) \to F(t_i) \to \lim F(t^n) \to \lim F(x^n)$ which sends $b \in F(t)$ in turn into b_i, then into $(b_n)_{n\in\mathbb{N}}$ and finally into $(a_n)_{n\in\mathbb{N}}$. This shows that the element $(a_n)_{n\in\mathbb{N}} \in \lim F(x^n) \subseteq \prod F(x^n)$ lifts to an element lying in $\lim \mathcal{T}(x^n, t)$ along the natural map $\prod \mathcal{T}(x_n, t) \to \prod F(x^n)$ which corresponds to b via the Yoneda isomorphism, and the proof of our claim is complete. □

Proposition 3.3.2. *Let $\mathcal{T}$ be a triangulated category with coproducts which is $\aleph_1$-perfectly generated by a projective class $(\mathcal{P}, \mathfrak{F})$. Let $F : \mathcal{T} \to \mathcal{A}b$ be a functor wich sends coproducts inti products. If every category $\mathcal{P}^{*n}/F$ has a weak terminal object, then F is representable.*

Proof. The conclusion follows from dual of Theorem 2.2.3 corroborated with Proposition 3.3.1. □

We will say that $\mathcal{T}$ is $\aleph_1$-perfectly generated by a set if it is $\aleph_1$-perfectly generated by a the projective class induced by that set, in the sense above. Thus Proposition above together with Lemma 3.2.17 give:

Theorem 3.3.3. *Let $\mathcal{T}$ be a triangulated category with coproducts which is $\aleph_1$-perfectly generated by a set. Then $\mathcal{T}$ is deconstructible and satisfies Brown representability.*

Remark 3.3.4. Our condition $\mathcal{T}$ to be $\aleph_1$-perfectly generated by a set is obviously equivalent to the hypothesis of [43, Theorem A].

Obviously every well–generated triangulated category in the sense of Neeman [60] is $\aleph_1$-perfectly generated by a set, in the sense above, as we may seen in [42], hence we obtain:

Corollary 3.3.5. *If $\mathcal{T}$ is a well–generated triangulated category then $\mathcal{T}$ is deconstructible, therefore it satisfies Brown representability.*

Chapter 4

Quasi-locally presentable categories

This chapter is based on [50] and contains an axiomatization of some properties satisfied by the abelianization of a well–generated triangulated categories. We also show that this structure allows us to derive Brown representability at this level from the general Freyd's adjoint functor theorem.

4.1 Quasi-locally presentable abelian categories

Denote by $\mathfrak{R}$ the class of all regular cardinals.

Consider a regular cardinal λ. A non–empty category $\mathcal{S}$ is called λ*–filtered* if the following two conditions are satisfied:

F1. For every set $\{s_i \mid i \in I\}$ of less that λ objects of $\mathcal{S}$ there are an object $s \in \mathcal{S}$ and morphisms $s_i \to s$ in $\mathcal{S}$, for all $i \in I$.

F2. For every set $\{\sigma_i : s \to t \mid i \in I\}$ of less that λ morphisms in $\mathcal{S}$, there is a morphism $\tau : t \to u$ such that $\tau\sigma_i = \tau\sigma_j$, for all $i, j \in I$.

Let A be an object of a category $\mathcal{A}$. Then the functor $\mathcal{A}(A, -)$ preserves the colimit of a diagram $\mathcal{S} \to \mathcal{A},\ s \mapsto X(s)$ in $\mathcal{A}$ (indexed over a category $\mathcal{S}$), if and only if every map $g : A \to \varinjlim_{s\in\mathcal{S}} X(s)$ factors as

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "g"] & \\ X(u) \arrow[r, "\xi_u"'] & \varinjlim_{s\in\mathcal{S}} X(s) \end{tikzcd}$$

through some of the canonical maps ξ_u with $u \in \mathcal{S}$, and every such factorization is essentially unique, in the sense that if $f_1, f_2 : A \to X(u)$ with $\xi_u f_1 = g = \xi_u f_2$

then there is $\sigma : u \to t$ a map in $\mathcal{S}$ such that $X(\sigma)f_1 = X(\sigma)f_2$. The object $A \in \mathcal{A}$ is called *λ–presentable* if $\mathcal{A}(A, -)$ preserves all λ–filtered colimits. The category $\mathcal{A}$ is called *locally λ–presentable* provided that it is cocomplete, and has a set $\mathcal{S}$ of λ–presentable objects such that every $X \in \mathcal{A}$ is a λ–filtered colimit of objects in $\mathcal{S}$ (see [1, Definition 1.17], but also [1, Remark 1.21])). Note that, if $\mathcal{A}$ is locally λ–presentable, then the subcategory $\mathcal{A}^\lambda$ of all λ–presentable objects in $\mathcal{A}$ is essentially small, and for every object $A \in \mathcal{A}$, the category $\mathcal{A}^\lambda/A$ is λ–filtered and

$$A \cong \varinjlim_{(X,\xi)\in\mathcal{A}^\lambda/A} X,$$

as we can see from [1, Proposition 1.22]. A category is called *locally presentable* if it is locally λ–presentable for some regular cardinal λ.

Remark 4.1.1. Let $\mathcal{A}$ be a locally λ–presentable category. Observe than the category $\mathcal{A}^o$ satisfies the hypotheses of Freyd's special adjoint functor theorem: it is well powered, complete and has a cogenerator (since the coproduct of all λ-presentable objects is a generator for $\mathcal{A}$). In particular, every contravariant functor $F : \mathcal{A} \to \mathcal{A}b$ which sends colimits into limits is representable. Indeed, we can view F as a covariant functor $\mathcal{A}^o \to \mathcal{A}b$ which must be representable, having a left adjoint. Let us write $F \cong \mathcal{A}(-, A)$, for some $A \in \mathcal{A}$. Thus the categories $\mathcal{A}^\lambda/A$ and $\mathcal{A}^\lambda/F$ are isomorphic, so

$$F \cong \mathcal{A}(-, \varinjlim_{(X,x)\in\mathcal{A}^\lambda/F} X).$$

We consider a cocomplete category $\mathcal{A}$ which is a union

$$\mathcal{A} = \bigcup_{\lambda\in\mathfrak{R}} \mathcal{A}_\lambda,$$

of a chain of subcategories $\{\mathcal{A}_\lambda \mid \lambda \in \mathfrak{R}\}$ such that $\mathcal{A}_\kappa \subseteq \mathcal{A}_\lambda$ for all $\kappa \leq \lambda$ and the subcategory $\mathcal{A}_\lambda$ locally λ–presentable and closed under colimits in $\mathcal{A}$, for any $\lambda \in \mathfrak{R}$. Denote by $I_\lambda : \mathcal{A}_\lambda \to \mathcal{A}$ the inclusion functor, which preserves colimits by our assumption. Note that by Freyd's special adjoint functor theorem, the subcategory $\mathcal{A}_\lambda$ is coreflective, that is I_λ has a right adjoint $R_\lambda : \mathcal{A} \to \mathcal{A}_\lambda$. We call *quasi-locally presentable* a category $\mathcal{A}$ as above satisfying the additional property that R_λ preserves colimits for all $\lambda \in \mathfrak{R}$. For such a quasi-locally presentable category $\mathcal{A}$ and a regular cardinal λ we denote by $\mathcal{A}_\lambda^\lambda$ the subcategory of all λ–presentable objects of $\mathcal{A}_\lambda$, which has to be skeletally small.

Lemma 4.1.2. *In a quasi-locally presentable category $\mathcal{A}$ it holds $\mathcal{A}_\kappa^\kappa \subseteq \mathcal{A}_\lambda^\lambda$, for every $\kappa \leq \lambda$.*

Proof. With the notations above, fix two cardinals $\kappa \leq \lambda$. Observe that if we denote $I_{\kappa,\lambda} : \mathcal{A}_\kappa \to \mathcal{A}_\lambda$ the inclusion functor, then it has a right adjoint namely $R_{\kappa,\lambda} = R_\kappa I_\lambda$. Since R_κ preserves colimits, $R_{\kappa,\lambda}$ satisfies the same property. Then for $A \in \mathcal{A}_\kappa^\kappa$ and for a λ-filtered (hence also κ-filtered) diagram $(X_i)_{i\in I}$ in $\mathcal{A}_\lambda$ we have the following chain of isomorphisms, showing that $I_{\kappa,\lambda}(A)$ is λ-presentable:

$$\begin{aligned}\mathcal{A}_\lambda(I_{\kappa,\lambda}(A), \varinjlim X_i) &\cong \mathcal{A}_\kappa(A, R_{\kappa,\lambda}(\varinjlim X_i)) \cong \mathcal{A}_\kappa(A, \varinjlim R_{\kappa,\lambda}(X_i)) \\ &\cong \varinjlim \mathcal{A}_\kappa(A, R_{\kappa,\lambda}(X_i)) \cong \varinjlim \mathcal{A}_\lambda(I_{\kappa,\lambda}(A), X_i).\end{aligned}$$

□

As a first example of quasi-locally presentable categories we mention first the classical locally presentable ones. Clearly if $\mathcal{A}$ is locally κ–presentable for some regular cardinal κ, then it is also quasi-locally presentable, with $\mathcal{A} = 0$ for all regular cardinals $\lambda < \kappa$ and $\mathcal{A}_\lambda = \mathcal{A}$, for all $\lambda \geq \kappa$.

Lemma 4.1.3. *Let $F : \mathcal{A} \to \mathcal{A}b$ be a contravariant functor which sends colimits into limits, defined on a quasi-locally presentable, abelian category $\mathcal{A}$. Then for every regular cardinal κ, there is $\lambda \in \mathfrak{R}$, $\lambda \geq \kappa$ such that*

$$FI_\kappa \cong \varinjlim_{(X,x)\in\mathcal{A}_\lambda^\lambda/F} \mathcal{A}(I_\kappa(-), X)$$

Proof. For any $\lambda \in \mathfrak{R}$, consider the coreflective locally λ–presentable subcategory

$$I_\lambda : \mathcal{A}_\lambda \leftrightarrows \mathcal{A} : R_\lambda$$

coming from the definition of a quasi-locally presentable category.

Fix $\kappa \in \mathfrak{R}$. For a skeleton $\mathcal{C}_0$ of $\mathcal{A}_\kappa^\kappa$, denote $C_0 = \coprod_{(U,u)\in\mathcal{C}_0/F} U$. Let λ be a regular cardinal such that

$$\lambda > \kappa + \operatorname{card}\mathcal{C}_0 + \sum_{U\in\mathcal{C}_0} \operatorname{card} F(U) + \sum_{U\in\mathcal{C}_0} \operatorname{card}\mathcal{A}(U, C_0) + \aleph_1.$$

Since $F : \mathcal{A} \to \mathcal{A}b$ sends colimits into limits, the same property is also true for $FI_\lambda : \mathcal{A}_\lambda \to \mathcal{A}b$. By Remark 4.1.1 we obtain $FI_\lambda \cong \mathcal{A}_\lambda(-, F_\lambda)$ for some $F_\lambda \in \mathcal{A}_\lambda$ satisfying

$$F_\lambda = \varinjlim_{(X,x)\in\mathcal{A}_\lambda^\lambda/F} X = \varinjlim_{(X,\xi)\in\mathcal{A}_\lambda^\lambda/F_\lambda} X,$$

with the canonical maps $\gamma_{(X,x)} : X \to F_\lambda$. Note that $\gamma_{(X,x)}$ is the image of (X, x) under the isomorphism of categories $\mathcal{A}_\lambda^\lambda/FI_\lambda \to \mathcal{A}_\lambda^\lambda/F_\lambda$. We have to show that

$$F(A) \cong \varinjlim_{(X,x)\in\mathcal{A}_\lambda^\lambda/F} \mathcal{A}(A, X),$$

for all $A \in \mathcal{A}_\kappa$. Since $A = I_\kappa(A) = I_\lambda(A)$ this means precisely that $\mathcal{A}(A, -)$ preservers the colimit of the diagram $\mathcal{A}_\lambda^\lambda/F \to \mathcal{A}$, $(X, x) \mapsto X$. In order to prove this, consider in the first step that A is a coproduct of objects in $\mathcal{A}_\kappa^\kappa$. Without losing the generality we may assume that $A = \coprod_{i\in I} U_i$, for some set I, and some $U_i \in \mathcal{C}_0$. Denote by $j_i : U_i \to A$, $(i \in I)$ the canonical injections. Let $g : A \to F_\lambda$ be a map in $\mathcal{A}$. Since for all $U \in \mathcal{C}_0$ we have $U \in \mathcal{A}_\kappa \subseteq \mathcal{A}_\lambda$, we may identify $\mathcal{C}_0/F$ with $\mathcal{C}_0/F_\lambda$ thus $C_0 = \coprod_{(U,v)\in\mathcal{C}_0/F_\lambda} U$ with the canonical injections $\epsilon_{(U,v)} : U \to C_0$. Since $gj_i \in \mathcal{A}(U_i, F_\lambda)$ we get a unique $f : A \to C_0$, such that $fj_i = \epsilon_{(U_i, gj_i)}$ from the universal property of the coproduct. Put $c_0 = (v)_{(U,v)\in\mathcal{C}_0/F_\lambda}$. We know by Lemma 4.1.2 that $\mathcal{A}^\kappa_\kappa \subseteq \mathcal{A}^\lambda_\lambda$, so the condition $\lambda > \sum_{U\in\mathcal{C}_0} \operatorname{card} F(U)$ assures us that $(C_0, c_0) \in \mathcal{A}^\lambda_\lambda/F_\lambda$. It follows $(C_0, c_0) \in \mathcal{A}^\lambda_\lambda/F$. Moreover by construction $\gamma_{(C_0,c_0)} f = g$, so g factors through $\gamma_{(C_0,c_0)}$, where γ is the canonical transformation above.

It remains to show that this factorization is essentially unique. Consider therefore two maps $f_1, f_2 : A \to C_0$ such that $\gamma_{(C_0,c_0)} f_1 = g = \gamma_{(C_0,c_0)} f_2$. Denote $\mathcal{N} = \{(U, h) \mid U \in \mathcal{C}_0, h \in \mathcal{A}(U, C_0) \text{ with } \gamma_{(C_0,c_0)} h = 0\}$, where $\mathcal{C}$ is a skeleton of $\mathcal{A}^\lambda_\lambda$, and put $C_1 = \coprod_{(U,h)\in\mathcal{N}} U$ with the canonical injections $k_{(U,h)} : U \to C_1$. By the choice of λ we have $\lambda > \operatorname{card} \mathcal{A}(U, C_0) \geq \operatorname{card} \mathcal{N}$, hence $(C_1, 0) \in \mathcal{A}^\lambda_\lambda/F$. We may even consider $(C_1, 0) \in \mathcal{C}/F$. We have $(U_i, (f_1 - f_2)j_i) \in \mathcal{N}$, hence there is a unique $\theta : A \to C_1$ such that $\theta j_i = k_{(U_i,(f_1-f_2)j_i)}$ for all $i \in I$. Further there is a unique morphism $\eta : C_1 \to C_0$ such that $\eta k_{(U,h)} = h$ for all $(U, h) \in \mathcal{N}$. Clearly η is a map in $\mathcal{A}^\lambda_\lambda/F$ between $(C_1, 0)$ and (C_0, c_0). If C is defined by the exactness of the sequence $C_1 \xrightarrow{\eta} C_0 \xrightarrow{\delta} C \to 0$, then $C \in \mathcal{A}^\lambda_\lambda$, because $\mathcal{A}^\lambda_\lambda$ is closed under cokernels (see [1, Proposition 1.16]). Since F sends cokernels into kernels, we infer that there is $c \in F(C)$ such that $F(\delta)(c) = c_0$. Thus $\delta : (C_0, c_0) \to (C, c)$ lies in $\mathcal{A}^\lambda_\lambda/F$, and $\delta(f_1 - f_2) = \delta\eta\theta = 0$, finishing the proof of the first step above.

Finally an arbitrary $A \in \mathcal{A}_\kappa$ is a colimit of objects in $\mathcal{A}^\kappa_\kappa$, so it is a cokernel of the form $A_1 \to A_0 \to A \to 0$ with A_1 and A_0 being coproducts of objects in $\mathcal{A}^\kappa_\kappa$. Using the first step before, we get easily

$$F(A) \cong \mathcal{A}(A, F_\lambda) \cong \underrightarrow{\operatorname{colim}}_{(X,x)\in\mathcal{A}^\lambda_\lambda/F} \mathcal{A}(A, X)$$

canonically. □

Remark 4.1.4. With the notations made in Lemma 4.1.3 and its proof, the argument used to show the fact that $\mathcal{A}(A, F_\lambda) \cong \underrightarrow{\operatorname{colim}}_{(X,x)\in\mathcal{A}^\lambda_\lambda/F} \mathcal{A}(A, X)$, for $A = \coprod_{i\in I} U_i$, with $U_i \in \mathcal{A}^\kappa_\kappa$ is inspired by [21, Lemma 2.11]. However, we didn't only change the settings, but we also improved the proof of Franke. A simple translation of his argument in our settings would require the condition $\operatorname{card} \mathcal{A}(U, X) \leq \lambda$ for all $U \in \mathcal{A}^\kappa_\kappa$ and all $X \in \mathcal{A}^\lambda_\lambda$. A priori is not clear how we may choose such a regular cardinal λ. Instead of this, we required $\sum_{U\in\mathcal{C}_0} \operatorname{card} \mathcal{A}(U, C_0) < \lambda$, where the left hand side of this inequality doesn't depend of λ.

Recall that we call *cofinal* a subcategory $\mathcal{S}$ of a category $\mathcal{C}$ satisfying the following two properties: For every $c \in \mathcal{C}$ there is a map $c \to s$ in $\mathcal{C}$ for some $s \in \mathcal{S}$; and for any two maps $c \to s_1$ and $c \to s_2$ in $\mathcal{C}$, with $s_1, s_2 \in \mathcal{S}$ there are $s \in \mathcal{S}$ and two maps $s_1 \to s$ and $s_2 \to s$ in $\mathcal{S}$ such that the composed morphisms $c \to s_1 \to s$ and $c \to s_2 \to s$ are equal. It is well–known that if $\mathcal{S}$ is a cofinal subcategory of $\mathcal{C}$, then colimits over $\mathcal{C}$ and colimits over $\mathcal{S}$ coincide (see [1, 0.11]).

Lemma 4.1.5. *Let $\mathcal{A}$ be an abelian category, and let $F : \mathcal{A} \to \mathcal{A}b$ be a contravariant, exact functor. Let $\mathcal{C} \subseteq \mathcal{A}$ be a subcategory closed under finite coproducts and cokernels. If $\mathcal{S}$ is a subcategory of $\mathcal{C}$ closed under finite coproducts and satisfying the property that every $X \in \mathcal{C}$ admits an embedding $0 \to X \to S$ into an object in $\mathcal{S}$, then $\mathcal{S}/F$ is a cofinal subcategory of $\mathcal{C}/F$.*

Proof. Let $(X, x) \in \mathcal{C}/F$. Consider an embedding $0 \to X \stackrel{\alpha}{\to} S$, with $S \in \mathcal{S}$. Thus $F(S) \stackrel{F(\alpha)}{\longrightarrow} F(X) \to 0$ is exact, showing that there exists $y \in F(S)$ with $F(\alpha)(y) = x$. Therefore α is a map in $\mathcal{C}/F$ between (X, x) and (S, y).

Now we claim that if $\alpha : X_1 \to X_2$ is a map in $\mathcal{C}$, and $x_2 \in F(X_2)$ is an element with the property $F(\alpha)(x_2) = 0$, then there is a morphism $\gamma \in \mathcal{C}/F((X_2, x_2), (S, y))$ into an object $(S, y) \in \mathcal{S}/F$ such that $\gamma\alpha = 0$. Indeed consider X being defined by exact sequence $X_1 \stackrel{\alpha}{\to} X_2 \stackrel{\beta}{\to} X \to 0$. Since the sequence of abelian groups $0 \to F(X) \stackrel{F(\beta)}{\to} F(X_2) \stackrel{F(\alpha)}{\to} F(X_1)$ is also exact and $F(\alpha)(x_2) = 0$, we obtain an element $x \in F(X)$ such that $F(\beta)(x) = x_2$. For obtaining the required γ, compose β with a morphism in $\mathcal{C}/F$ from (X, x) into an object (S, y), which is constructed as in the first part of this proof.

Finally for two morphisms

$$\alpha_1 \in \mathcal{C}/F((X, x), (S_1, y_1)) \text{ and } \alpha_2 \in \mathcal{C}/F((X, x), (S_2, y_2)),$$

denote by ρ_1 and ρ_2 the respective injections of the coproduct $S_1 \amalg S_2$. Then $F(\rho_1\alpha_1 - \rho_2\alpha_2)(y_1, y_2) = x - x = 0$, so our claim for $\alpha = \rho_1\alpha_1 - \rho_2\alpha_2$ gives a morphism $(S_1 \amalg S_2, (y_1, y_2)) \to (S, y)$ in $\mathcal{C}/F$, with $S \in \mathcal{S}$, such that the composed morphisms $X \to S_1 \to S_1 \amalg S_2 \to S$ and $X \to S_2 \to S_2 \amalg S_2 \to S$ are equal. □

Let $\kappa \in \mathfrak{R}$. As usually, a *κ–(co)product* means a (co)product of less that κ objects. We say that a quasi-locally presentable abelian category $\mathcal{A}$ is *weakly κ–generated* if $\mathcal{A}$ coincide with its smallest full subcategory containing $\mathcal{A}_\kappa$ and being closed under kernels, cokernels, extensions and κ–coproducts. We also need the following notation:

$$\mathrm{Inj}_\lambda\mathcal{A} = \{S \in \mathcal{A} \mid S \text{ is injective and } S \in \mathcal{A}_\lambda^\lambda\}.$$

Theorem 4.1.6. *Let $\mathcal{A}$ be a quasi-locally presentable, abelian category which is weakly κ–generated, for some regular cardinal κ. Suppose also that, for any regular cardinal $\lambda \geq \kappa$, every $X \in \mathcal{A}_\lambda^\lambda$ admits an embedding $0 \to X \to S$ into an object $S \in \mathrm{Inj}_\lambda\mathcal{A}$. Then every exact, contravariant functor $F : \mathcal{A} \to \mathcal{A}b$ which sends coproducts into products is representable (necessarily by an injective object).*

Proof. Fix a contravariant exact functor $F : \mathcal{A} \to \mathcal{A}b$, which sends coproducts into products. Consider the obvious natural transformation

$$\phi : \underrightarrow{\mathrm{colim}}_{(X,x)\in\mathcal{A}_\lambda^\lambda/F} \mathcal{A}(-, X) \to F.$$

Since F sends colimits into limits, Lemma 4.1.3 applies and tells us that there is $\lambda \in \mathfrak{R}$, $\lambda \geq \kappa$ such that ϕ restricts to an isomorphism:

$$\varinjlim_{(X,x)\in \mathcal{A}_\lambda^\lambda/F} \mathcal{A}(I_\kappa(-), X) \cong FI_\kappa.$$

We know that $\mathcal{A}_\lambda^\lambda/F$ is λ–filtered (see [26, Korollar 5.4]), hence colimits of abelian groups indexed over this category are exact and commute with products of less that λ objects (see [26, Satz 5.2]). Since every $X \in \mathcal{A}_\lambda^\lambda$ admits an embedding in an object $S \in \mathrm{Inj}_\lambda\mathcal{A}$, we deduce by Lemma 4.1.5 that $\mathrm{Inj}_\lambda\mathcal{A}/F$ is a cofinal subcategory of $\mathcal{A}_\lambda^\lambda/F$, so

$$\varinjlim_{(X,x)\in \mathcal{A}_\lambda^\lambda/F} \mathcal{A}(-, X) \cong \varinjlim_{(S,s)\in \mathrm{Inj}_\lambda\mathcal{A}/F} \mathcal{A}(-, S)$$

is an exact functor. We infer that the full subcategory of $\mathcal{A}$ consisting of all objects A for which ϕ_A is an isomorphism contains $\mathcal{A}_\kappa$ and is closed under kernels, cokernels, extensions and κ-coproducts (since $\lambda \geq \kappa$). Therefore it is equal to $\mathcal{A}$ forced by the hypothesis of weak κ–generation. This means that ϕ is a natural isomorphism, hence a skeleton of $\mathcal{A}_\lambda^\lambda$ forms a solution set for F. We conclude that F is representable by the general Freyd's adjoint functor theorem. □

Example 4.1.7. The following example shows that the conclusion of Theorem 4.1.6 requires a kind of weak generation.

Recall that an abelian category is called *locally Grothendieck* if every set of objects may be included in subcategory which is Grothendieck (see [76]). Let K be a field. The category $\mathcal{A} = \bigcup_{\lambda\in\mathfrak{R}} \mathrm{Mod}(K^\lambda)$ considered in [76] is locally Grothendieck. Note that there it is shown that the category $\bigcup_{\lambda\in\mathfrak{R}} \mathrm{Mod}(K^\lambda)$ is example of a category of λ-pure global dimension greater that 1, for all $\lambda \in \mathfrak{R}$; this fact is related to Brown representability. Here by $\mathrm{Mod}(K^\lambda)$ we denote the category of right modules over the ring K^λ. The category $\mathcal{A}$ is also quasi-locally presentable. Indeed it is a a big union of a chain of Grothendieck (hence locally presentable) subcategories $\mathcal{A}_\lambda = \mathrm{Mod}(K^\lambda)$. For all $\kappa \leq \lambda$ in $\mathfrak{R}$ we have $K^\kappa = K^\lambda e$, where $e = e(\kappa, \lambda) \in K^\lambda$ is a central idempotent defined by $e_\gamma = 1$ for $\gamma \leq \kappa$ and 0 otherwise. Thus K^κ is a direct summand of K^λ, and all $X \in \mathrm{Mod}(K^\lambda)$ decomposes as $X = Xe \oplus X(1-e)$. Moreover for $X, Y \in \mathrm{Mod}(K^\lambda)$ there is no nonzero homomorphisms between Xe and $Y(1-e)$, hence we have

$$\mathrm{Hom}_{K^\lambda}(X, Y) = \mathrm{Hom}_{K^\kappa}(Xe, Ye) \oplus \mathrm{Hom}_{K^\lambda(1-e)}(X(1-e), Y(1-e)).$$

Thus we can see $\mathrm{Mod}(K^\kappa)$ as a full split subcategory of $\mathrm{Mod}(K^\lambda)$. We deduce that for every fixed $\kappa \in \mathfrak{R}$ and for every $X \in \mathcal{A}$, there is $\lambda \geq \kappa$ such that $X \in \mathrm{Mod}(K^\lambda)$. The assignment $X \mapsto Xe$, where $e = e(\kappa, \lambda)$ induces a well defined functor $R_\kappa : \mathcal{A} \to \mathrm{Mod}(K^\kappa)$ which is both the left and the right adjoint of the inclusion functor I_κ; this follows by the fact that $\mathrm{Mod}(K^\kappa)$ is a full split subcategory of $\mathrm{Mod}(K^\lambda)$. Thus both the inclusion functor $\mathrm{Mod}(K^\kappa)$ and its right adjoint preserve colimits.

Then we can construct a non-representable exact contravariant functor $F : \mathcal{A} \to \mathcal{A}b$, which sends coproducts into products (the same idea will be used in the future in the proof of Proposition 5.1.2). For every $\lambda \in \mathfrak{R}$, denote by λ^+ the successor of λ and consider Q_{λ^+} to be an injective cogenerator of $\mathrm{Mod}(K^{\lambda^+})$. The K^{λ^+}-module $Y_\lambda = Q_{\lambda^+}(1-e)$, where $e = e(\lambda, \lambda^+)$, is injective and satisfies $\mathrm{Hom}_{K^{\lambda^+}}(X, Y_\lambda) = 0$ for all $X \in \mathrm{Mod}(K^\lambda)$. The contravariant functor

$$F : \mathcal{A} \to \mathcal{A}b, F(X) = \prod_{\lambda \in \mathfrak{R}} \mathcal{A}(X, Y_\lambda)$$

is well defined. In fact, for $X \in \mathrm{Mod}(K^\kappa)$, we have $\mathcal{A}(X, Y_\lambda) = 0$ if $\lambda \geq \kappa$, hence $F(X) = \prod_{\lambda < \kappa} \mathcal{A}(X, Y_\lambda)$. Obviously F is exact and sends coproducts into products. But F is not representable, since the strict inclusion of $\mathrm{Mod}(K^\lambda)$ into $\mathrm{Mod}(K^{\lambda^+})$ implies that the cogenerator Q_{λ^+} must contain a nonzero part Y_λ in $\mathrm{Mod}(K^{\lambda^+}(1-e))$. The representability of F would means the existence of the product $Y = \prod_{\lambda \in \mathfrak{R}} Y_\lambda$ in $\mathcal{A}$. But this is absurd since Y would have a proper class of endomorphisms, and such objects don't exist in $\mathcal{A}$.

On the other hand we have:

Proposition 4.1.8. *Consider the above locally Grothendieck category*

$$\mathcal{A} = \bigcup_{\lambda \in \mathfrak{R}} \mathrm{Mod}(K^\lambda).$$

A contravariant functor $F : \mathcal{A} \to \mathcal{A}b$ is representable if and only if it sends colimits into limits and there is $\kappa \in \mathfrak{R}$ such that $F \cong F I_\kappa R_\kappa$.

Proof. If $F \cong \mathcal{A}(-, Y)$ for some $Y \in \mathcal{A}$ then there is $\kappa \in \mathfrak{R}$ such that $Y \in \mathrm{Mod}(K^\kappa)$. Thus for every $X \in \mathcal{A}$, there is $\lambda \geq \kappa$ such that $X \in \mathrm{Mod}(K^\lambda)$, hence $F(X) = \mathcal{A}(X, Y) \cong \mathcal{A}(Xe, Y) \cong F I_\kappa R_\kappa(X)$.

Conversely if F sends colimits into limits then, as in the proof of Lemma 4.1.3, we obtain $F I_\kappa \cong \mathrm{Hom}_{K^\kappa}(-, Y)$, for some $Y \in \mathrm{Mod}(K^\kappa)$. Combining this with $F \cong F I_\kappa R_\kappa$ we deduce:

$$F \cong \mathrm{Hom}_{K^\kappa}(R_\kappa(-), Y) \cong \mathcal{A}(-, I_\kappa(Y)),$$

therefore F is representable. □

Example 4.1.9. In Theorem 4.1.6 the exactness of the functor $F : \mathcal{A} \to \mathcal{A}b$ (which sends coproducts into products) is an essential hypothesis. More precisely, the weaker requirement that F sends colimits into limits is not sufficient to conclude that it is representable. For showing this, suppose that the quasi-locally presentable category $\mathcal{A}$ from the Theorem 4.1.6 is abelian (as in the motivating case of the next Section) but is not locally presentable, that is $\mathcal{A} \neq \mathcal{A}_\lambda$ for every $\lambda \in \mathfrak{R}$. The fact that $\mathcal{A}$ is weakly generated which is used in combination with the exactness of F doesn't play any role in this example. The exactness of R_λ implies that $\mathcal{A}_\lambda$ is equivalent to quotient category of $\mathcal{A}$ modulo the Serre subcategory $\operatorname{Ker} R_\lambda = \{X \in \mathcal{A} \mid R_\lambda(X) = 0\}$. But R_λ is not an equivalence, forcing $\operatorname{Ker} R_\lambda \neq 0$. Consider $0 \neq X_\lambda \in \mathcal{A}$ such that $R_\lambda(X_\lambda) = 0$, for every $\lambda \in \mathfrak{R}$. Strictly speaking we need here a version of axiom of choice which works for proper classes. As in Example 4.1.7, we infer that the functor

$$F = \prod_{\lambda \in \mathfrak{R}} \mathcal{A}(-, X_\lambda)$$

is well defined since for every $X \in \mathcal{A}$ we have $X \in \mathcal{A}_\kappa$ for some $\kappa \in \mathfrak{R}$, so $\mathcal{A}(X, X_\lambda) = 0$ for all $\lambda \geq \kappa$. It is easy to see that this functor does the job we claim.

4.2 The abelianization of a well generated triangulated category is quasi-locally presentable

A category $\mathcal{C}$ is called λ–*cocomplete* if $\mathcal{C}$ has λ–coproducts (that is, coproducts of fewer than λ objects) and cokernels. It is easy to see that $\mathcal{C}$ is λ–cocomplete if and only if it contains all colimits of diagrams with less that λ morphisms. A $\mathcal{C}$-module over a λ–cocomplete category is called λ–*left exact* if it is left exact and sends λ–coproducts into products. Provided that the category $\mathcal{C}$ is essentially small, the class $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ is actually a set for all $\mathcal{C}$-modules X, Y. Thus we are allowed to consider the category $\operatorname{Mod}(\mathcal{C})$ of all $\mathcal{C}$-modules. If $\mathcal{C}$ is also λ–cocomplete, then denote by $\operatorname{Lex}_\lambda(\mathcal{C}^o, \mathcal{A}b)$ the full subcategory of $\operatorname{Mod}(\mathcal{C})$ consisting of λ–left exact modules. We know that $\operatorname{Lex}_\lambda(\mathcal{C}^o, \mathcal{A}b)$ is a locally λ–presentable category, and the embedding $\mathcal{C} \to \operatorname{Lex}_\lambda(\mathcal{C}^o, \mathcal{A}b)$ given by $X \mapsto \mathcal{C}(-, X)$ identifies $\mathcal{C}$, up to isomorphism, with the subcategory of λ–presentable objects in $\operatorname{Lex}_\lambda(\mathcal{C}^o, \mathcal{A}b)$ (see [26, Korollar 7.9]).

As before, let λ denote a regular cardinal. If $\mathcal{S}$ is an preadditive, essentially small category with λ–coproducts, denote by $\operatorname{Ex}_\lambda(\mathcal{S}^o, \mathcal{A}b)$ the full subcategory of $\operatorname{Mod}(\mathcal{S})$, consisting of those modules which preserve λ–products. Clearly a finitely presentable $\mathcal{S}$-module, that is an element in $\operatorname{mod}(\mathcal{S})$, preserves arbitrary products, hence it belongs to $\operatorname{Ex}_\lambda(\mathcal{S}^o, \mathcal{A}b)$.

Lemma 4.2.1. *For a regular cardinal λ, consider an additive, essentially small category $\mathcal{S}$ having λ–coproducts. Then $\operatorname{Ex}_\lambda(\mathcal{S}^o, \mathcal{A}b)$ is a locally λ–presentable category, and the embedding $\operatorname{mod}(\mathcal{S}) \xrightarrow{\subseteq} \operatorname{Ex}_\lambda(\mathcal{S}^o, \mathcal{A}b)$ identifies $\operatorname{mod}(\mathcal{S})$ with the full subcategory of $\operatorname{Ex}_\lambda(\mathcal{S}^o, \mathcal{A}b)$ consisting of all λ–presentable objects.*

Proof. The category $\operatorname{mod}(\mathcal{S})$ has obviously λ–coproducts and cokernels, so it is λ–cocomplete. According to [44, Lemma B.1], there is an equivalence of categories

$$\operatorname{Lex}_\lambda(\operatorname{mod}(\mathcal{S})^o, \mathcal{A}b) \to \operatorname{Ex}_\lambda(\mathcal{S}^o, \mathcal{A}b),\ X \mapsto XH_{\mathcal{S}},$$

where $H_{\mathcal{S}} : \mathcal{S} \to \operatorname{mod}(\mathcal{S})$ denotes the Yoneda functor. Thus $\operatorname{Ex}_\lambda(\mathcal{S}^o, \mathcal{A}b)$ is locally λ–presentable. Further, the identification of λ–presentable objects in $\operatorname{Ex}_\lambda(\mathcal{S}^o, \mathcal{A}b)$ follows by discussion above concerning λ–presentable objects in $\operatorname{Lex}_\lambda(\mathcal{C}^o, \mathcal{A}b)$. $\square$

Suppose now that $\mathcal{T}$ is well κ–generated triangulated category, having a perfectly generating set $\mathcal{S}$ consisting of κ-small objects (see Definition 1.3.4). Consider the subcategory $\mathcal{T}^\lambda$ of all λ–compact objects. Clearly $\mathcal{T}^\lambda$ is skeletally small and has λ-coproducts. Denote $\mathrm{mod}_\lambda(\mathcal{T}) = \mathrm{Ex}_\lambda((\mathcal{T}^\lambda)^o, \mathcal{A}b)$, for $\lambda \geq \kappa$ and $\mathrm{mod}_\lambda(\mathcal{T}) = 0$ otherwise. We know by [60, Proposition A.1.8] that $\mathrm{mod}_\lambda(\mathcal{T})$ is locally λ–presentable, and the functor by [60, Proposition 6.5.3] that it may be identified to a coreflective subcategory of $\mathrm{mod}(\mathcal{T})$ via the fully faithful functor $I_\lambda : \mathrm{mod}_\lambda(\mathcal{T}) \to \mathrm{mod}(\mathcal{T})$ which is the left adjoint of the restriction functor $R_\lambda : \mathrm{mod}(\mathcal{T}) \to \mathrm{mod}_\lambda(\mathcal{T})$.

Proposition 4.2.2. *Fix a regular cardinal $\kappa > \aleph_0$. Provided that $\mathcal{T}$ is a compactly κ–generated triangulated category, then* $\mathrm{mod}(\mathcal{T})$ *is a quasi-locally presentable abelian category which is weakly κ–generated.*

Proof. Denote by $\mathcal{A}$ the smallest subcategory of $\mathrm{mod}(\mathcal{T})$ which is closed under kernels, cokernels, extensions, countable coproducts and contains mod_κ-$\mathcal{T}$. Let us show that $\mathrm{mod}(\mathcal{T}) = \mathcal{A}$. Observe first that if $T \to U \to X \to Y \to Z$ is an exact sequence with $T, U, Y, Z \in \mathcal{A}$ then we can construct the commutative diagram with exact rows and column

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
T \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d, equal] & X' \arrow[r] \arrow[d] & 0 & \\
T \arrow[r] & U \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, equal] & Z \arrow[d, equal] \\
 & 0 \arrow[r] & X'' \arrow[r] \arrow[d] & Y \arrow[r] & Z \\
 & & 0 & &
\end{tikzcd}$$

showing that $X \in \mathcal{A}$. Therefore if $x \to y \to z \to \Sigma x$ is a triangle in $\mathcal{T}$ with $H(x), H(z) \in \mathcal{A}$ then $H(y) \in \mathcal{A}$. It is shown in Theorem 3.2.10 that every object $x \in \mathcal{T}$ is isomorphic to a homotopy colimit of a tower $x^0 \to x^1 \to \cdots$ such that $x^0 = 0$ and for every $n \in \mathbb{N}$ we have a triangle $p_n \to x^n \to x^{n+1} \rightsquigarrow$ with p_n being a coproduct of objects in $\mathcal{T}^\kappa$. Inductively $H(x^n) \in \mathcal{A}$, for all $n \in \mathbb{N}$, hence $H(\coprod_{n\in\mathbb{N}} x^n) \cong \coprod_{n\in\mathbb{N}} H(x^n) \in \mathcal{A}$, and finally $H(x) \in \mathcal{A}$. Now, for every $X \in \mathrm{mod}(\mathcal{T})$ there is an exact sequence $H(y) \to H(x) \to X \to 0$, with $x, y \in \mathcal{T}$, thus $X \in \mathcal{A}$.

Note that we have already shown that $\mathcal{T}$ coincides with its smallest $\aleph_1$-localizing subcategory which contains a skeleton of $\mathcal{T}^\kappa$. Therefore the proof of [60, Proposition 8.4.2] (more precisely [60, 8.4.2.3]) works for our case, hence $\mathcal{T} = \bigcup_{\lambda\geq\kappa} \mathcal{T}^\lambda$, and further $\mathrm{mod}(\mathcal{T}) = \bigcup_{\lambda\in\mathfrak{R}} \mathrm{mod}_\lambda(\mathcal{T})$. In addition an immediate consequence of Lemma [60, 6.5.1] is that the right adjoint of the inclusion functor $\mathrm{mod}_\lambda(\mathcal{T}) \to \mathrm{mod}(\mathcal{T})$ preserves colimits, and all conditions from the definition of a weakly κ-generated quasi-locally presentable category are fulfilled. $\square$

Theorem 4.2.3. *If $\mathcal{T}$ is a well–generated triangulated category, then every functor $F : \mathrm{mod}(\mathcal{T}) \to \mathcal{A}b$ which is contravariant, exact and sends coproducts into products is representable.*

Proof. Without losing the generality we may assume that $\mathcal{T}$ is well κ–generated, for some $\kappa \geq \aleph_1$ (if not, we replace κ by $\aleph_1$). By Proposition 4.2.2, $\mathrm{mod}(\mathcal{T})$ is a weakly κ–generated quasi-locally presentable category. In order to apply Theorem 4.1.6 we have only to show that every λ–presentable object X of $\mathrm{mod}_\lambda(\mathcal{T})$ admits an embedding into an object in $S \in \mathrm{mod}_\lambda(\mathcal{T})$ which is λ–presentable in $\mathrm{mod}_\lambda(\mathcal{T})$ and injective in $\mathrm{mod}(\mathcal{T})$. But this follows immediately from Lemma 4.2.1, since, according to [60, Corollary 5.1.23], every $X \in \mathrm{mod}(\mathcal{T}^\lambda)$ admits an embedding into an object of the form $H(x)$ with $x \in \mathcal{T}^\lambda$. $\square$

Note that the category $\mathrm{mod}(\mathcal{T})$ is usually “huge”, in the sense that it is not well (co)powered, as we learned on [60, Appendix C]. Thus Proposition 4.2.2 and Theorem 4.2.3 provide an example of such a huge category which is quasi-locally presentable and for which representability Theorem 4.1.6 applies.

Combining Theorems 4.2.3 and 2.1.1 we obtain (once again) a new proof for:

Corollary 4.2.4. *Every well–generated triangulated categories satisfies Brown representability.*

Chapter 5

Homotopy category of complexes

In this chapter we characterize abelian categories $\mathcal{A}$ for which $\mathbf{K}(\mathcal{A})$ and/or $\mathbf{K}(\mathcal{A})^o$ satisfy Brown representability. The results were first published in [56] and [51].

5.1 Homtopy categories satisfying Brown representability

The category $\mathcal{T}$ is called *locally well–generated* (in the sense of [74, Definition 3.1]) if for any set $\mathcal{S}$ (not a proper class!) of objects of $\mathcal{T}$, $\mathrm{Loc}(\mathcal{S})$ is well–generated.

Let $\mathcal{T}$ be a triangulated category and denote by $\mathfrak{R}$ the (proper) class of all infinite regular cardinal numbers. In what follows we often need an increasing chain

$$\mathcal{S}_{\aleph_0} \subseteq \mathcal{S}_{\aleph_1} \subseteq \mathcal{S}_{\aleph_2} \subseteq \cdots \subseteq \mathcal{S}_{\kappa} \subseteq \dots$$

of skeletally small triangulated subcategories of $\mathcal{T}$ indexed by $\mathfrak{R}$ such that the union $\bigcup_{\kappa \in \mathfrak{R}} \mathcal{S}_{\kappa}$ is the whole of $\mathcal{T}$.

Remark 5.1.1. Strictly speaking, it is not clear how to obtain such a chain in general using the axioms of ZFC alone. But there are two workarounds. First, if we work with a more concrete triangulated category, it may be possible to construct the chain directly. For example, if $\mathcal{T} = \mathbf{K}(\mathrm{Mod}(R))$, then $\mathcal{S}_{\kappa}$ can be defined as the subcategory of all complexes formed by κ–presented modules. Second, if we insist on general $\mathcal{T}$, we can adopt some suitable axiomatization of set theory which allows us to well-order the universe of all sets (eg. the von Neumann–Bernays–Gödel set theory). Then we can easily construct the chain using the induced well-ordering of objects of $\mathcal{T}$. The same applies to the proof of Proposition 5.1.2 below, where we strictly speaking use the Axiom of Choice for proper classes.

Now we can formulate a simple but important obstruction to Brown representability.

Proposition 5.1.2. *Let $\mathcal{T}$ be a triangulated category with coproducts. Suppose that $\mathcal{T}$ possesses an increasing chain $(\mathcal{S}_\kappa \mid \kappa \in \mathfrak{R})$ of skeletally small triangulated subcategories such that $\mathcal{T} = \bigcup_{\kappa\in\mathfrak{R}} \mathcal{S}_\kappa$ and $\mathcal{S}_\kappa^\perp \neq 0$ for all $\kappa \in \mathfrak{R}$. Then $\mathcal{T}$ does not satisfy Brown representability.*

Dually, suppose $\mathcal{T}$ is triangulated, has products and has an increasing chain $\{\mathcal{S}_\kappa \mid \kappa \in \mathfrak{R}\}$ of skeletally small triangulated subcategories such that $\mathcal{T} = \bigcup_{\kappa\in\mathfrak{R}} \mathcal{S}_\kappa$ and ${}^\perp\mathcal{S}_\kappa \neq 0$ for all $\kappa \in \mathfrak{R}$. Then $\mathcal{T}^o$ does not satisfy Brown representability.

Proof. We prove only the first part, the other is dual. Choose for each $\kappa \in \mathfrak{R}$ an object $0 \neq Y_\kappa \in \mathcal{S}_\kappa^\perp$. We consider the functor

$$F = \prod_{\kappa\in\mathfrak{R}} \mathcal{T}(-, Y_\kappa) : \mathcal{T} \longrightarrow \mathcal{A}b.$$

Note that F is a well-defined functor. Indeed, recall that any $X \in \mathcal{T}$ is contained in $\mathcal{S}_\kappa$ for some $\kappa \in \mathfrak{R}$, so $\mathcal{T}(X, Y_\lambda) = 0$ for all $\lambda \geq \kappa$ and the product defining FX is essentially set-indexed. Moreover, F is homological and sends coproducts to products.

Now we are essentially done, since if F were represented by some object in $\mathcal{T}$, it would have to be the product of $\{Y_\kappa \mid \kappa \in \mathfrak{R}\}$ in $\mathcal{T}$, which cannot exist. To give a formal argument, assume for the moment that there is some $Y \in \mathcal{T}$ and a natural equivalence

$$\eta : \mathcal{T}(-, Y) \longrightarrow F.$$

For each $\kappa \in \mathfrak{R}$ we then have an idempotent natural transformation $\epsilon_\kappa : F \to F$ given as the composition

$$F \longrightarrow \mathcal{T}(-, Y_\kappa) \longrightarrow F$$

which, by the Yoneda lemma, induces an idempotent morphism $e_\kappa : Y \to Y$ in $\mathcal{T}$. Since $(\epsilon_\kappa \mid \kappa \in \mathfrak{R})$ is a proper class of pairwise orthogonal non-zero idempotent endotransformations of F, the collection $\{e_\kappa \mid \kappa \in \mathfrak{R}\}$ would have to be a proper class of endomorphisms of Y with the same properties. This is absurd since $\mathcal{T}(Y, Y)$ is a set. $\square$

Now we are ready to give a proof of Theorem 5.1.3. For a more concrete construction of a non-representable functor $\mathbf{K}(\mathcal{A}b) \to \mathcal{A}b$, see Example 5.2.8 below.

Theorem 5.1.3. *Let $\mathcal{T}$ be a locally well–generated triangulated category. Then $\mathcal{T}$ satisfies Brown representability if and only if $\mathcal{T}$ is well–generated. In particular, if R is a ring which is not right pure semisimple, for instance $R = \mathbb{Z}$, then $\mathbf{K}(\mathrm{Mod}(R))$ does not satisfy Brown representability.*

Proof. If $\mathcal{T}$ is well–generated, or equivalently $\mathcal{T} = \mathrm{Loc}(\mathcal{S})$ for some set $\mathcal{S}$, then Brown representability holds by [60, Proposition 8.4.2]. Let us, therefore, assume that $\mathcal{T}$ is not well–generated.

As discussed above, we have an increasing chain $\{\mathcal{S}_\kappa \mid \kappa \in \mathfrak{R}\}$ of skeletally small triangulated subcategories of $\mathcal{T}$ such that $\mathcal{T} = \bigcup_{\kappa\in\mathfrak{R}} \mathcal{S}_\kappa$. Let us put $\mathcal{L}_\kappa = \mathrm{Loc}(\mathcal{S}_\kappa)$; by definition each $\mathcal{L}_\kappa$ is well generated and our assumption ensures $\mathcal{L}_\kappa \subsetneqq \mathcal{T}$. It follows from [60, 9.1.13 and 9.1.19] that each $X \in \mathcal{T}$ admits a triangle

$$\Gamma_\kappa X \longrightarrow X \longrightarrow L_\kappa X \longrightarrow \Gamma_\kappa \Sigma X \tag{$*$}$$

with $\Gamma_\kappa X \in \mathcal{L}_\kappa$ and $L_\kappa X \in \mathcal{S}_\kappa^\perp$. So given arbitrary $X \in \mathcal{T} \setminus \mathcal{L}_\kappa$ it follows $0 \neq L_\kappa X \in \mathcal{S}_\kappa^\perp$. Now we just apply Proposition 5.1.2.

Finally, the second part of the theorem follows from [74, 2.6 and 3.5]: If R is a ring which is not right pure semi-simple, $\mathbf{K}(\mathrm{Mod}(R))$ is locally well generated but not well generated. □

5.2 Brown representability for the dual of a homotopy category

Fix the additive category $\mathcal{A}$. In this section we discuss Brown representability for $\mathbf{K}(\mathcal{A})^o$. For an object $G \in \mathcal{A}$ we denote by $\mathrm{Prod}(G)$ respectively $\mathrm{Add}(G)$ the full subcategory consisting of direct factors (or equivalently, direct summands) of a product (respectively coproduct) of copies of G (assuming that the requested products or coproducts exist).

Definition 5.2.1. We say that $\mathcal{A}$ has a *product generator* if there is an object $G \in \mathcal{A}$ such that $\mathcal{A} = \mathrm{Prod}(G)$. For the dual situation when $\mathcal{A} = \mathrm{Add}(G)$ we use the more standard terminology $\mathcal{A}$ *is pure semisimple* (see [73, Definition 2.1 and Proposition 2.2]).

Based on [57, Definition 2.24], we introduce the following concept:

Definition 5.2.2. Let $\mathcal{A}$ be an additive category and $\mathcal{X}$ a full subcategory. Given $M \in \mathcal{A}$, by an *augmented proper right $\mathcal{X}$-resolution* we understand a complex of the form

$$X_M : \qquad \ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow M \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots,$$

such that $X^i \in \mathcal{X}$ for all $i \geq 0$ and $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X_M, X'[n]) = 0$ for all $X' \in \mathcal{X}$ and $n \in \mathbb{Z}$.

A favorable fact is that such resolutions often do exist.

Lemma 5.2.3. *Let $\mathcal{A}$ be an additive category with products and splitting idempotents, let $X \in \mathcal{A}$ and put $\mathcal{X} = \mathrm{Prod}(X)$. Then any $M \in \mathcal{A}$ admits an augmented proper right $\mathcal{X}$-resolution $X_M \in \mathbf{K}(\mathcal{A})$. Moreover, $X_M = 0$ in $\mathbf{K}(\mathcal{A})$ if and only if $M \in \mathcal{X}$.*

Remark 5.2.4. The lemma is true also without $\mathcal{A}$ having splitting idempotents, but we keep the assumption for the sake of simplicity.

Proof. We will construct the terms X^i of an augmented resolution

$$X_M : \qquad \dots \longrightarrow 0 \longrightarrow 0 \longrightarrow M \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} \cdots$$

by induction on i. We put $X^0 = X^{\mathrm{Hom}_{\mathcal{A}}(M,X)}$ and take for d^{-1} the obvious morphism. Having constructed X^i for $i \geq 0$, we set

$$\mathcal{Z}_i = \{f \in \mathrm{Hom}_{\mathcal{A}}(X^i, X) \mid f \circ d^{i-1} = 0\},$$

Then we can take $X^{i+1} = X^{\mathcal{Z}_i}$ and construct $d^i : X^i \to X^{i+1}$ in the obvious way.

For the second part, assume that $X_M = 0$ in $\mathbf{K}(\mathcal{A})$, so it is a contractible complex. In particular, $d^{-1} : M \to X^0$ splits, so $M \in \mathrm{Prod}(X) = \mathcal{X}$. The other implication is easy. □

Now we show a consequence of non-existence of a product generator for $\mathcal{A}$, which is important in connection with Proposition 5.1.2.

Proposition 5.2.5. *Let $\mathcal{A}$ be an additive category with products and splitting idempotents. If $\mathcal{A}$ does not have a product generator, then ${}^\perp\mathcal{S} \neq 0$ in $\mathbf{K}(\mathcal{A})$ for every set (not a proper class!) $\mathcal{S} \subseteq \mathbf{K}(\mathcal{A})$.*

Proof. Suppose $\mathcal{A}$ has no product generator and $\mathcal{S} \subseteq \mathbf{K}(\mathcal{A})$ is a set of complexes. Let $\mathcal{U} \subseteq \mathcal{A}$ be the set of all objects occurring in the components of complexes in $\mathcal{S}$, and let $\mathcal{X} = \mathrm{Prod}(\mathcal{U})$. Then clearly $\mathcal{S} \subseteq \mathbf{K}(\mathcal{X})$, so it suffices to show that ${}^\perp\mathbf{K}(\mathcal{X}) \neq 0$ in $\mathbf{K}(\mathcal{A})$.

To this end, we have $\mathcal{X} \subsetneqq \mathcal{A}$ since $\mathcal{A}$ has no product generator. Thus, we can take an object $M \in \mathcal{A} \setminus \mathcal{X}$ and construct, using Lemma 5.2.3, an augmented proper right $\mathcal{X}$–resolution X_M of M such that $X_M \neq 0$ in $\mathbf{K}(\mathcal{A})$. We would like to see that $X_M \in {}^\perp\mathbf{K}(\mathcal{X})$, but this has been proved by Murfet in [57, Proposition 2.27] (using crucially the fact that X_M is a complex which is bounded below). □

Theorem 5.2.6. *Let $\mathcal{A}$ be an additive category with products. If $\mathbf{K}(\mathcal{A})^o$ satisfies Brown representability, then $\mathcal{A}$ has a product generator. In particular $\mathbf{K}(\mathcal{A}b)^o$ does not satisfy Brown representability.*

Proof. First of all, we may without loss of generality assume that $\mathcal{A}$ has splitting idempotents. If not, we replace $\mathcal{A}$ by its idempotent completion $\tilde{\mathcal{A}}$ (see e.g. [4, §1]). Since $\mathbf{K}(\mathcal{A})$ has splitting idempotents by [60, Proposition 1.6.8 and Remark 1.6.9], it follows that the inclusion $\mathbf{K}(\mathcal{A}) \subseteq \mathbf{K}(\tilde{\mathcal{A}})$ is a triangle equivalence.

Next we suppose that $\mathcal{A}$ has no product generator and prove the existence of a non-representable homological product-preserving functor $F : \mathbf{K}(\mathcal{A}) \to \mathcal{A}b$. Namely, we choose an increasing chain

$$\mathcal{S}_{\aleph_0} \subseteq \mathcal{S}_{\aleph_1} \subseteq \mathcal{S}_{\aleph_2} \subseteq \cdots \subseteq \mathcal{S}_\kappa \subseteq \dots$$

of skeletally small triangulated subcategories of $\mathbf{K}(\mathcal{A})$ indexed by $\mathfrak{R}$ such that the union $\bigcup_{\kappa\in\mathfrak{R}}\mathcal{S}_\kappa$ is the whole of $\mathbf{K}(\mathcal{A})$ (cf. Remark 5.1.1). Then, however, Proposition 5.2.5 ensures that ${}^\perp\mathcal{S}_\kappa \neq 0$ in $\mathbf{K}(\mathcal{A})$ for each $\kappa \in \mathfrak{R}$, and so we are in the situation of Proposition 5.1.2, which asserts the existence of such a functor.

Finally, we must prove that $\mathcal{A}b$ has no product generator. For this purpose, let us fix a prime number $p \in \mathbb{N}$. Using the notation from [25, §XI.65], we define inductively for every abelian group G and every ordinal σ:

$$p^\sigma G = \begin{cases} G, & \text{if } \sigma = 0 \\ p(p^{\sigma-1}G), & \text{if } \sigma \text{ is non limit.} \\ \bigcap\limits_{\rho<\sigma} p^\rho G, & \text{if } \sigma \text{ is limit.} \end{cases}$$

The *p-length* $l(G)$ of the group G is then by definition the minimum ordinal λ such that $p^{\lambda+1}G = p^\lambda G$. Note that for any family $(G_i \mid i \in I)$ of abelian groups, we have the formula

$$l\big(\prod G_i\big) = \sup\big\{l(G_i) \mid i \in I\big\}.$$

Thus, to prove that $\mathcal{A}b$ has no product generator, it suffices to construct abelian groups of arbitrary length. However, such families of groups are known. For instance Walker's groups P_β [78] (whose construction can also be found in [60, Construction C.2.1]) or generalized Prüfer groups [25, pp. 85–86]. □

Remark 5.2.7. The non-existence of a product generator for $\mathrm{Mod}(R)$ seems to be a much more widespread phenomenon. If $X \in \mathrm{Mod}(R)$ is a product generator, then $\mathrm{Ext}^1_R(M, X) = 0$ implies that M is projective for each $M \in \mathrm{Mod}(R)$. That is, X is a *projective test module* in the sense of [20, p. 408]. If R is not right perfect it is, however, consistent with ZFC + GCH that there are no projective test modules and in particular no product generators.

We conclude the section with more concrete examples of non-representable (co)homological functors $\mathbf{K}(\mathcal{A}b) \to \mathcal{A}b$.

Example 5.2.8. Let us for each $\kappa \in \mathfrak{R}$ denote by $\mathcal{A}_\kappa$ the full subcategory of $\mathcal{A}b$ formed by all groups of cardinality smaller than κ, and put $\mathcal{T} = \mathbf{K}(\mathcal{A}b)$.

If we take for a given κ a group P_κ of length $\kappa + 1$ (e.g. Walker's group P_κ from [78]), then clearly $P_\kappa \notin \mathrm{Prod}(\mathcal{A}_\kappa)$, since the length of any group from $\mathrm{Prod}(\mathcal{A}_\kappa)$ is at most κ. Thus, recalling the arguments above, we see that the augmented proper right $\mathrm{Prod}(\mathcal{A}_\kappa)$–resolution of P_κ, which we denote by Y_κ, is nonzero in $\mathbf{K}(\mathcal{A}b)$ and belongs to ${}^\perp\mathbf{K}(\mathrm{Prod}(\mathcal{A}_\kappa))$. In particular, the functor

$$F = \prod_{\kappa\in\mathfrak{R}} \mathcal{T}(Y_\kappa, -) : \mathcal{T} \longrightarrow \mathcal{A}b,$$

is a well-defined homological functor which sends products in $\mathcal{T}$ to products of abelian groups, but it is not representable by an object of $\mathcal{T}$.

Let us also explicitly construct a contravariant non-representable functor. In fact, we will use the formally dual statement to Theorem 5.2.6 and its proof for this rather than Theorem 5.1.3. The key point is [17, Theorem 3.1] by Chase, which implies that for any uncountable $\kappa \in \mathfrak{R}$, we have $\mathbb{Z}^\kappa \notin \mathrm{Add}(\mathcal{A}_\kappa)$. Here, $\mathrm{Add}(\mathcal{A}_\kappa)$ denotes as usual the closure of $\mathcal{A}_\kappa$ in $\mathcal{A}b$ under taking direct sums and summands. Therefore, denoting by Y'_κ the augmented proper left $\mathrm{Add}(\mathcal{A}_\kappa)$–resolution of $\mathbb{Z}^\kappa$ (with the obvious meaning), we can infer exactly as before that the functor

$$F' = \prod_{\kappa \in \mathfrak{R}} \mathcal{T}(-, Y'_\kappa) : \mathcal{T} \longrightarrow \mathcal{A}b,$$

is a well-defined cohomological functor which sends coproducts in $\mathcal{T}$ to products of abelian groups, but it is not representable by an object of $\mathcal{T}$.

Lemma 5.2.9. *Let $\mathcal{A}$ be an additive category with split idempotents and products, which possesses a product generator G. Denote $\mathcal{S} = \{\Sigma^n G \mid n \in \mathbb{Z}\}$ the closure of G under suspensions and desuspensions in $\mathbf{K}(\mathcal{A})$.*

a) If given two composable maps $X \to Y \to Z$ whose composition is 0 in $\mathcal{A}$, then $X \to Y$ factors through a subobject $Y' \leq Y$ such that the composed map $Y' \to Y \to Z$ vanishes, then $\mathbf{K}(\mathcal{A})$ is $\mathcal{S}$-cofiltered.

b) If $\mathcal{A}$ has images or kernels, then $\mathbf{K}(\mathcal{A})$ is $\mathcal{S}$-cofiltered.

Proof. a) We will show inductively that a bounded complex with less than $n+1$ non–zero entries is in $\mathrm{Prod}_n(\mathcal{S})$, where n runs over all positive integers. This is clear for $n = 0$, since G is a product generator of $\mathcal{A}$. Now we suppose the property true for any complex with $\leq n$ non–zero entries. Let

$$\cdots \to 0 \to X^0 \to \cdots \to X^n \to 0 \to \cdots$$

be a bounded complex. The diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X^n \arrow[r] \arrow[d, "="] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X^0 \arrow[r] \arrow[d, "="] & \cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[d, "="] & X^n \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & X_0 \arrow[r] & \cdots \arrow[r] & X^{n-1} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is an exact sequence of complexes which splits in each degree. According to [35, Example 6.1] it leads to a triangle proving the induction step.

Finally consider an infinite complex

$$X = \cdots \longrightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \longrightarrow \cdots.$$

By hypothesis, the map d^{n-1} factors through a subobject $Y^n \leq X^n$, such that $Y^n \longrightarrow X^n \xrightarrow{d^n} X^{n+1}$ vanishes, for all $n \in \mathbb{Z}$. For all $i \in \mathbb{N}$, consider the bounded complex

$$X(i) = \cdots \to 0 \to Y^{-i} \to X^{-i} \to X^{-i+1} \to \cdots \to X^{i-1} \longrightarrow X^i \to 0 \to \cdots ,$$

and the map of complexes $\epsilon(i) : X(i+1) \to X(i)$ as in the following diagram:

$$\begin{tikzcd}
 & \cdots \arrow[r] & 0 \arrow[r] & Y^{-i} \arrow[r] & X^{-i} \arrow[r] & \cdots \arrow[dddl, bend left] \\
\cdots \arrow[r] & Y^{-i-1} \arrow[r] \arrow[ur] & X^{-i-1} \arrow[r] \arrow[ur] & X^{-i} \arrow[r] \arrow[ur, equal] & \cdots \arrow[dl, bend left] & \\
 & \cdots & X^{i+1} \arrow[l] \arrow[dr] & X^{i} \arrow[l] \arrow[dr, equal] & & \\
 & & \cdots & 0 \arrow[l] & X^{i} \arrow[l] &
\end{tikzcd}$$

As in [34, Lemma 2.6] we infer that X is isomorphic in $\mathbf{K}(\mathcal{A})$ to the homotopy limit of a the chain of bounded complexes

$$\cdots \longrightarrow X(2) \xrightarrow{\epsilon(1)} X(1) \xrightarrow{\epsilon(0)} X(0),$$

thus X is $\mathcal{S}$-cofiltered.

b) We apply a) with $Y^n = \operatorname{im} d^{n-1}$ or $Y^n = \ker d^n$, for all $n \in \mathbb{Z}$. □

Theorem 5.2.10. *Let $\mathcal{A}$ be an additive category with products and split idempotents, possessing also images or kernels. Then $\mathbf{K}(\mathcal{A})^o$ satisfies Brown representability if and only if $\mathcal{A}$ has a product generator. In particular, if R is a ring then $\mathbf{K}(\mathrm{Mod}(R))^o$ satisfies Brown representability if and only if* $\mathrm{Mod}(R)$ *has a product generator.*

Proof. The direct implication is Theorem 5.2.6, whereas the converse follows by Lemma 5.2.9 b) and Theorem 3.1.3. Finally note that the category $\mathrm{Mod}(R)$ is additive with products and has both images and kernels. □

Remark 5.2.11. If the ring R is pure semisimple, then $\mathrm{Mod}(R) = \mathrm{Add}(G)$ for some $G \in \mathrm{Mod}(R)$ (in fact G is the direct sum of a family of representatives of all isomorphism classes of finitely presentable modules). In this case, $\mathrm{Add}(G)$ is closed under products, so G is product–complete hence $\mathrm{Add}(G) = \mathrm{Prod}(G)$ (see [41, Theorem 6.7]). Consequently $\mathbf{K}(\mathrm{Mod}(R))^o$ satisfies Brown representability, by Theorem above. This was already known since $\mathrm{Mod}(R)$ is a pure semisimple finitely presentable category which is closed under products, so it is compactly generated by [73, Theorem 5.2]. It would be therefore interesting to characterize the class of rings R for which the module category $\mathrm{Mod}(R)$ has a product generator. If we could indicate a non pure semisimple ring belonging to this class, then we would produce an example of a triangulated category with products and coproducts, namely $\mathcal{K} = \mathbf{K}(\mathrm{Mod}(R))$ such that $\mathcal{K}^o$ satisfies Brown representability, but $\mathcal{K}$ don't. To the best of our knowledge, such an example is yet unknown. Note that, according to [8], pure semisimplicity condition is equivalent to the existence of a product generator, provided that we assume some extra set theoretical axiomes. This suggests that the equivalence between Brown representability for $\mathbf{K}(\mathrm{Mod}(R))$ and for $\mathbf{K}(\mathrm{Mod}(R))^o$ is not decidable in ZFC.

Remark 5.2.12. There is an isomorphism of categories $\mathbf{K}(\mathcal{A})^o \xrightarrow{\sim} \mathbf{K}(\mathcal{A}^o)$, which is easy to establish (for example, this is written down in [47, Theorem 2.1.1]). Applying this isomorphism of categories, we may dualize all results in this section. Thus we may conclude that if $\mathcal{A}$ is an additive category with split idempotents and coproducts, possessing also images or cokernels, then $\mathbf{K}(\mathcal{A})$ satisfies Brown representability theorem if and only if $\mathcal{A}$ is pure semisimple. Note that this statement is already known for $\mathcal{A} = \mathrm{Mod}(R)$, or more generally for a finitely accessible category with coproducts $\mathcal{A}$, as we may see by a combination between [56, Theorem 1] and [73, Proposition 2.6]. However the results in [56] and [73] may not be dualized in order to obtain Theorem 5.2.10 back, since the argument used there for showing that $\mathbf{K}(\mathcal{A})$ satisfies Brown representability, where $\mathcal{A}$ is a pure semisimple, finitely accessible additive category with coproducts goes as follows: If $\mathcal{A}$ enjoys all these properties, then $\mathbf{K}(\mathcal{A})$ is well generated by [73, Theorm 5.2], therefore it satisfies Brown representability by [60, Theorem 8.3.3 and proposition 8.4.2]. But none of the notions "finitely accessible category" and "well generated triangulated category" is self–dual.

Remark 5.2.13. Let R be a ring with $\mathrm{gl.dim}\, R \leq 1$. Then the category $\mathrm{Inj}(R)$ of all injective modules is additive, closed under products, idempotents and images and every injective cogenerator of $\mathrm{Mod}(R)$ is a product generator for $\mathrm{Inj}(R)$. Thus Theorem 5.2.10 gives another proof for the fact that $\mathbf{K}(\mathrm{Inj}(R))^o$ satisfies Brown representability. This fact is already known since $\mathbf{K}(\mathrm{Inj}(R))$ is equivalent to the derived category which is compactly generated.

5.3 Functors without adjoints

Given a ring R, a right R-module M is called *Mittag–Leffler* if the canonical map of groups

$$M \otimes_R \left(\prod_{i \in I} N_i \right) \longrightarrow \prod_{i \in I} (M \otimes_R N_i)$$

is injective for each family of left R-modules $\{N_i \mid i \in I\}$. This concept comes from [71].

Let $\mathcal{D}$ be the class of all flat Mittag–Leffler R-modules. There are several characterizations of modules in $\mathcal{D}$ already in work of Raynaud and Gruson [71], but the latest one is due to Herbera and Trlifaj, [29, Theorem 2.9]: Flat Mittag-Leffler modules coincide with so called $\aleph_1$-projective modules. For $R = \mathbb{Z}$, this simply means that $G \in \mathcal{D}$ if and only if each countable subgroup of G is free, which is a special case of [3, Proposition 7] proved by Azumaya and Facchini.

Theorem 5.3.1. *Let R be a countable ring and let $\mathcal{D}$ be the class of all right flat Mittag–Leffler R-modules. Then $\mathbf{K}(\mathcal{D})$ is always closed under coproducts in $\mathbf{K}(\mathrm{Mod}(R))$, but the inclusion functor $\mathbf{K}(\mathcal{D}) \to \mathbf{K}(\mathrm{Mod}(R))$ has a right adjoint if and only if R is a right perfect ring. In particular, a right adjoint does not exist for $R = \mathbb{Z}$.*

Proof. It is rather easy to see that $\mathcal{D}$ is closed under direct sums and contains all projective modules.

Assume first that R is right perfect. Then $\mathcal{D}$ coincides with the class of projective modules. In particular, $\mathbf{K}(\mathcal{D})$ is a well-generated triangulated category by [62, Theorem 1.1], so the inclusion $\mathbf{K}(\mathcal{D}) \to \mathbf{K}(\mathrm{Mod}(R))$ has a right adjoint by [60, Theorem 8.4.4].

On the other hand, assume that $\mathbf{K}(\mathcal{D}) \to \mathbf{K}(\mathrm{Mod}(R))$ has a right adjoint. Given any $G \in \mathrm{Mod}(R)$ and considering it as a stalk complex in degree 0, we have the counit of adjunction $\varepsilon_G : D \to G$. Let us take the R–module homomorphism $f = \varepsilon_G^0 : D^0 \to G$ in degree 0. Clearly $D^0 \in \mathcal{D}$ and it is easy to see that any R–module homomorphism $f' : D' \to G$ with $D' \in \mathcal{D}$ factors through f. That is, $\mathcal{D}$ is what is usually called a precovering class in $\mathrm{Mod}(R)$. However, according to [5, Theorem 6], this implies for a countable ring R that it is right perfect. $\square$

Example 5.3.2. Theorem 5.3.1 gives an example of a triangulated coproduct preserving functor which has no right adjoint, namely the inclusion functor $\mathbf{K}(\mathcal{D}) \to \mathbf{K}(\mathcal{A}b)$, where $\mathcal{D}$ is the full subcategory of all flat Mittag–Leffler abelian groups. Using the equivalence of categories $\mathbf{K}(\mathcal{D})^o \xrightarrow{\sim} \mathbf{K}(\mathcal{D}^o)$ from Remark 5.2.12, we obtain a triangulated product preserving functor which has no left adjoint.

Here we will provide another example of this kind, which holds only in an extension of ZFC. More precisely, assume there are no measurable cardinals. For every cardinal λ let us denote by $\mathbb{Z}^\lambda$ the product of λ-copies of $\mathbb{Z}$ and by $\mathbb{Z}^{<\lambda}$ its subgroup consisting of sequences with support (i.e. the set of non-zero entries) of cardinality smaller then λ. Let $\mathcal{A} \subseteq \mathcal{A}b$ be the closure under products and direct factors of the class of all abelian groups of the form $\mathbb{Z}^\lambda/\mathbb{Z}^{<\lambda}$, where λ runs over all regular cardinals. The inclusion functor $\mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}b)$ is triangulated and preserves products. If we suppose that it has a left adjoint then $\mathbf{K}(\mathcal{A})$ must be preenveloping in $\mathbf{K}(\mathcal{A}b)$ by Corollary 2.2.5. For $X \in \mathcal{A}b$, the complex having X in degree 0 and 0 elsewhere must have an $\mathbf{K}(\mathcal{A})$-preenvelope, which is a complex A with entries in $\mathcal{A}$. As in the proof of Theorem 5.3.1, it is not hard to see that $X \to A^0$ is an $\mathcal{A}$-preenvelope of X. But this contradicts [15, Proposition 2.5], where it is shown that, under the hypothesis of nonexistence of measurable cardinals, the class $\mathcal{A}$ is not preenveloping in $\mathcal{A}b$.

Chapter 6

Brown representability for the dual

In this chapter we show that the following categories: $\mathbf{D}(\mathcal{A})^o$ for a wide class of abelian categories $\mathcal{A}$; $\mathbf{K}(\mathrm{Proj}(R))^o$ for an arbitary ring with several objects R and $\mathbf{K}(\mathrm{Proj}(R, Q))^o$, where $\mathrm{Proj}(R, Q)$ is the category of projective R-representations of a quiver Q satisfy Brown representability. The material is taken from [54], [52] and [53], but [52] is almost entirely replaced by the more general [53].

6.1 The dual of Brown representability for some derived categories

In this section our triangulated category is $\mathcal{T} = \mathbf{D}(\mathcal{A})$, the derived category of an abelian category $\mathcal{A}$. For a complex $X^\bullet$ and a positive integer $n \in \mathbb{N}$, consider the truncation

$$X^{\geq -n} = (0 \to \mathrm{B}^{-n}(X^\bullet) \to X^{-n} \to X^{-n+1} \to \cdots).$$

There is a map of complexes $X^{\geq -(n+1)} \to X^{\geq -n}$ which is the identity $X^i \to X^i$ in degrees $i \geq -n$, the zero map in degrees $i < -(n+1)$ and the canonical epimorphism $X^{-(n+1)} \to \mathrm{B}^{-n}(X^\bullet)$ in degree $-(n+1)$. In this way, we obtain an inverse tower

$$X^{\geq 0} \leftarrow X^{\geq -1} \leftarrow X^{\geq -2} \leftarrow \cdots .$$

Following [65], the category $\mathbf{D}(\mathcal{A})$ is said to be *left–complete*, provided that it has products and with the notation above $X^\bullet \cong \underleftarrow{\mathrm{holim}}\, X^{\geq -n}$.

An example of a non–left–complete derived category can be found in [65]. In counterpart, some examples of left–complete categories will be provided later.

Theorem 6.1.1. *Let $\mathcal{A}$ be a complete abelian category possessing an injective cogenerator, and let $\mathbf{D}(\mathcal{A})$ be its derived category. If $\mathbf{D}(\mathcal{A})$ is left–complete, then $\mathbf{D}(\mathcal{A})$ has small hom–sets and $\mathbf{D}(\mathcal{A})^o$ satisfies Brown representability.*

Before we prove Theorem 6.1.1 we state some immediate consequences. Recall that a complete abelian category $\mathcal{A}$ is said to be *AB*4^*-*n*, with $n \in \mathbb{N}$, if the i-th derived functor of the direct product functor is zero, for all $i > n$ (see also [70] or [30]). Clearly AB4*-0 categories are the same as AB4* categories, that is abelian categories with exact products.

Corollary 6.1.2. *Let $\mathcal{A}$ be an abelian complete category possessing an injective cogenerator. If $\mathcal{A}$ is AB4*-n, for some $n \in \mathbb{N}$ and $\mathbf{D}(\mathcal{A})$ has products, then $\mathbf{D}(\mathcal{A})$ has small hom–sets and $\mathbf{D}(\mathcal{A})^o$ satisfies Brown representability.*

Proof. We know by [30, Theorem 1.3], that $\mathbf{D}(\mathcal{A})$ is left–complete, hence Theorem 6.1.1 applies. □

Let $\mathcal{A}$ be an abelian category with enough injectives. An *injective resolution* of $X \in \mathcal{A}$ is a complex of injectives $E^\bullet$ which is zero in negative degrees, together with an augmentation map $X \to E^\bullet$, such that the complex $0 \to X \to E^0 \to E^1 \to \cdots$ is acyclic. The *injective dimension* of an object $X \in \mathcal{A}$ is defined to be the smallest $n \in \mathbb{N}$ for which X has an injective resolution of the form

$$0 \to X \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to 0,$$

or ∞ if such an injective resolution does not exist. Equivalently, X has injective dimension n if it is the smallest non–negative integer for which $\mathrm{Ext}^{n+1}(-, X)$ vanishes. The *global injective dimension* of $\mathcal{A}$ is defined to be the supremum of all injective dimensions of its objects.

Remark 6.1.3. Products in module categories are exact, that is $\mathrm{Mod}(R)$ is AB4* for every ring R (with or without one), hence Corollary 6.1.2 applies. But in this case the derived category is known to be compactly generated, hence both $\mathbf{D}(\mathcal{A})$ and $\mathbf{D}(\mathcal{A})^o$ satisfy Brown representability, for example by [43, Theorem A and Theorem B]. An example of a Grothendieck AB4* category which has no nonzero projectives, hence it is not equivalent to a module category, may be found in [70, Section 4]. Note also that in [30, Theorem 1.1] there are other examples of abelian categories $\mathcal{A}$ which are AB4*-n, for some $n \in \mathbb{N}$, that is categories for which $\mathbf{D}(\mathcal{A})^o$ satisfies Brown representability, by Corollary 6.1.2 above.

Corollary 6.1.4. *Let $\mathcal{A}$ be an abelian complete category possessing an injective cogenerator. If $\mathcal{A}$ is of finite global injective dimension and $\mathbf{D}(\mathcal{A})$ has products, then $\mathbf{D}(\mathcal{A})$ has small hom–sets and $\mathbf{D}(\mathcal{A})^o$ satisfies Brown representability.*

Proof. We want to apply Corollary 6.1.2, so we will to show that $\mathcal{A}$ is AB4*-n, where n is the global injective dimension of $\mathcal{A}$. Fix an index set I. The k-th derived functor of the product $\prod^{(k)} : \mathcal{A}^I \to \mathcal{A}$ can be computed as follows: Consider arbitrary objects $X_i \in \mathcal{A}$ with $i \in I$. For every i choose an injective resolution $X_i \to E_i^\bullet$ of length less than or equal to n. Then $\prod^{(k)} X_i = \mathrm{H}^k(\prod E_i^\bullet)$, therefore $\prod^{(k)} X_i = 0$ for $k > n$. □

Corollary 6.1.5. *If $\mathcal{A}$ is the category of quasi-coherent sheaves over a quasi-compact and separated scheme then $\mathbf{D}(\mathcal{A})$ has small hom–sets and $\mathbf{D}(\mathcal{A})^o$ satisfies Brown representability. In particular, the conclusion holds for the category $\mathcal{A}$ of quasi-coherent sheaves over $\mathbb{P}^d_R$, where $\mathbb{P}^d_R$ is the projective d-space, $d \in \mathbb{N}^*$, over an arbitrary commutative ring with one R.*

Proof. The category of quasi–coherent sheaves is Grothendieck, hence $\mathbf{D}(\mathcal{A})$ satisfies Brown representability (see for example [2, Theorem 5.8]). Consequently, $\mathbf{D}(\mathcal{A})$ has products, by [60, Proposition 8.4.6]. Moreover, according to [30, Remark 3.3], the category of quasi–coherent sheaves over a quasi–compact, separated scheme is AB4*-n, for some $n \in \mathbb{N}$.

Finally, $\mathbb{P}^d_R$ is obtained by gluing together $d+1$ affine open sets (see [77, 4.4.9]). Hence, it is quasi–compact (see also exercise [77, 5.1.D]). Moreover, $\mathbb{P}^d_R$ is separated by [77, Proposition 10.1.5]. □

The proof of Theorem 6.1.1 is based on a combination between Theorem 3.1.3 and an adaptation of the argument in [37, Appendix]. For performing it, fix a complete abelian category $\mathcal{A}$, which has an injective cogenerator.

Recall that a complex $X^\bullet \in \mathbf{K}(\mathcal{A})$ is called *homotopically injective* if

$$\mathbf{K}(\mathcal{A})(N^\bullet, X^\bullet) = 0,$$

for any acyclic complex $N^\bullet$. Denote by $\mathbf{K}_i(\mathcal{A})$ the full subcategory of $\mathbf{K}(\mathcal{A})$ consisting of homotopically injective complexes. It follows immediately, that $\mathbf{K}_i(\mathcal{A})$ is a triangulated subcategory of $\mathcal{A}$ closed under products and direct summands. Dually, we can define the *homotopically projective* complexes and we write $\mathbf{K}_p(\mathcal{A})$ for the full subcategory of $\mathbf{K}(\mathcal{A})$ consisting of such complexes. A *homotpically injective resolution* of a complex $X^\bullet \in \mathbf{K}(\mathcal{A})$ is by definition a quasi-isomorphism $X^\bullet \to E^\bullet$, with $E^\bullet$ homotopically injective. Homotopically injective and projective complexes and resolutions were first defined by Spaltenstein in [72], but we follow the approach in [37]. If every complex in $\mathbf{K}(\mathcal{A})$ has a homotopically injective (projective) resolution, then this resolution yields a left (right) adjoint of the inclusion functor $\mathbf{K}_i(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ (respectively $\mathbf{K}_p(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$); the argument in [37, 1.2] generalizes with no change in this more general case. For example, if R is a ring and $\mathcal{A} = \mathrm{Mod}(R)$ is the category of all right modules over R, then $\mathcal{A}$ has enough projective and enough injective objects, and by [37, 1.1. and 1.2] we have equivalences of categories

$$\mathbf{K}_p(\mathrm{Mod}(R)) \xrightarrow{\sim} \mathbf{D}(\mathrm{Mod}(R)) \xleftarrow{\sim} \mathbf{K}_i(\mathrm{Mod}(R)).$$

More generally, if $\mathcal{A}$ is a Grothendieck category, it may not have enough projectives, and the left side functor might not be an equivalence. But it must have enough injectives, and the right side equivalence must hold as it can be seen from [2, Section 5]. Another proof of this fact is contained in [21, Section 3].

We consider double complexes with entries in $\mathcal{A}$, whose differentials go from bottom to top and from left to right. That is, a double complex is a commutative diagram of the form:

$$X^{\bullet,\bullet} = \left(\begin{array}{ccc} X^{i+1,j} & \xrightarrow{d_h^{i+1,j}} & X^{i+1,j+1} \\ {\scriptstyle d_v^{i,j}}\uparrow & & \uparrow{\scriptstyle d_v^{i,j+1}} \\ X^{i,j} & \xrightarrow[d_h^{i,j}]{} & X^{i,j+1} \end{array} \right)_{i,j\in\mathbb{Z}}$$

such that $d_v^2 = 0 = d_h^2$. We denote by $X^{\bullet,j}$ the columns and by $X^{i,\bullet}$ the rows of $X^{\bullet,\bullet}$.

Let $X^\bullet \in \mathbf{C}(\mathcal{A})$ be a complex. We identify it with a double complex concentrated in the 0-th column, making explicit the reason for which simple complexes are columns. A *Cartan–Eilenberg injective resolution* for $X^\bullet$ (*CE injective resolution* for short) is a right half–plane double complex $E^{\bullet,\bullet}$ (that is $E^{i,j} = 0$ for $j < 0$), together with an augmentation map (of double complexes) $X^\bullet \to E^{\bullet,\bullet}$ (with the identification above) such that $E^{i,\bullet} = 0$ provided that $X^i = 0$ and the induced sequences

$$0 \to \mathrm{H}^i(X^\bullet) \to \mathrm{H}^i(E^{\bullet,0}) \to \mathrm{H}^i(E^{\bullet,1}) \to \cdots,$$

$$0 \to \mathrm{B}^i(X^\bullet) \to \mathrm{B}^i(E^{\bullet,0}) \to \mathrm{B}^i(E^{\bullet,1}) \to \cdots$$

are injective resolutions for all $i \in \mathbb{Z}$ (see [79, Definition 5.7.1]). If $X^\bullet \to E^{\bullet,\bullet}$ is a CE injective resolution, then the induced sequences

$$0 \to \mathrm{Z}^i(X^\bullet) \to \mathrm{Z}^i(E^{\bullet,0}) \to \mathrm{Z}^i(E^{\bullet,1}) \to \cdots,$$

$$0 \to X^i \to E^{i,0} \to E^{i,1} \to \cdots$$

are injective resolutions for all $i \in \mathbb{Z}$ (see [79, Exercise 5.7.1]). For constructing a CE injective resolution for a given complex $X^\bullet$ we start with injective resolutions for $\mathrm{H}^i(X^\bullet)$ and $\mathrm{B}^i(X^\bullet)$, for all $i \in \mathbb{Z}$. Since the sequences $0 \to \mathrm{B}^i(X^\bullet) \to \mathrm{Z}^i(X^\bullet) \to \mathrm{H}^i(X^\bullet) \to 0$ and $0 \to \mathrm{Z}^i(X^\bullet) \to X^i \to \mathrm{B}^{i+1}(X^\bullet) \to 0$ are short exact, we use horseshoe lemma in order to construct injective resolutions for $\mathrm{Z}^i(X^\bullet)$ and X^i. Assembling together these data we obtain the desired CE injective resolution is $X^\bullet \to E^{\bullet,\bullet}$ (see also [79, Lemma 5.7.1]). If $E^{\geq -n,\bullet}$ is the truncated double complex having the columns

$$E^{\geq -n,j} = (0 \to \mathrm{B}^{-n}(E^{\bullet,j}) \to E^{-n.j} \to E^{-n+1,j} \to \cdots)^t$$

then by the very definition of a CE injective resolution we infer that $X^{\geq -n} \to E^{\geq -n,\bullet}$ is a CE injective resolution for the truncated complex.

Remark 6.1.6. The sequences $0 \to \mathrm{B}^i(E^{\bullet,j}) \to \mathrm{Z}^i(E^{\bullet,j}) \to \mathrm{H}^i(E^{\bullet,j}) \to 0$ and $0 \to \mathrm{Z}^i(E^{\bullet,j}) \to E^{i,j} \to \mathrm{B}^{i+1}(E^{\bullet,j}) \to 0$ have injective components, hence they are split exact for all $i, j \in \mathbb{Z}, j \geq 0$.

Next we define the *cototalization* of a double complex $X^{\bullet,\bullet}$ with $X^{i,j} \in \mathcal{A}$ as the simple complex $\mathrm{Cot}(X^{\bullet,\bullet})$ having entries:

$$\mathrm{Cot}(X^{\bullet,\bullet})^n = \prod_{i+j=n} X^{i,j}$$

and whose differentials are induced by using the universal property of the product by the maps

$$\prod_{i+j=n} X^{i,j} \to X^{p,q-1} \times X^{p-1,q} \overset{(d_h^{p,q-1},d_v^{p-1,q})}{\longrightarrow} X^{p,q}$$

for all $p, q \in \mathbb{Z}$ with $p + q = n + 1$.

Lemma 6.1.7. *Consider a complete abelian category $\mathcal{A}$, which has an injective cogenerator. If $X^\bullet \to E^{\bullet,\bullet}$ is a CE injective resolution of the complex $X^\bullet \in \mathbf{C}(\mathcal{A})$ then $\mathrm{Cot}(E^{\bullet,\bullet}) \cong \underleftarrow{\mathrm{holim}}\mathrm{Cot}(E^{\geq -n,\bullet})$ is homotopically injective.*

Proof. For every $n \in \mathbb{N}$ we observe that there is a map of double complexes $E^{\geq -(n+1),\bullet} \to E^{\geq -n,\bullet}$ which is the identity $E^{i,\bullet} \to E^{i,\bullet}$ for $i \geq -n$, the zero map for $i < -(n+1)$ and the epimorphism $E^{-(n+1),\bullet} \to \mathrm{B}^{-n}(E^{\bullet,\bullet})$ for $i = -(n+1)$. Hence Remark 6.1.6 tells us that $E^{\geq -(n+1),\bullet} \to E^{\geq -n,\bullet}$ are split epimorphisms in each degree, for every $n \in \mathbb{N}$. According to [57, Lemma 2.17] they induce degree–wise split epimorphisms

$$\mathrm{Cot}(E^{\geq -(n+1),\bullet}) \to \mathrm{Cot}(E^{\geq -n,\bullet}),$$

for all $n \in \mathbb{N}$. Thus there is a degree–wise split short exact sequence in $\mathbf{C}(\mathcal{A})$

$$0 \to \underleftarrow{\lim}\mathrm{Cot}(E^{\geq -n,\bullet}) \to \prod_{n\in\mathbb{N}} \mathrm{Cot}(E^{\geq -n,\bullet}) \overset{1-shift}{\longrightarrow} \prod_{n\in\mathbb{N}} \mathrm{Cot}(E^{\geq -n,\bullet}) \to 0$$

which induces a triangle in $\mathbf{K}(\mathcal{A})$. On the other hand, we have

$$\underleftarrow{\lim}\mathrm{Cot}(E^{\geq -n,\bullet}) \cong \mathrm{Cot}(E^{\bullet,\bullet})$$

in $\mathbf{C}(\mathcal{A})$, and the induced triangle leads to an isomorphism

$$\underleftarrow{\mathrm{holim}}\mathrm{Cot}(E^{\geq -n,\bullet}) \cong \mathrm{Cot}(E^{\bullet,\bullet})$$

in $\mathbf{K}(\mathcal{A})$ (see also [34, Lemma 2.6]). As we noticed, $\mathbf{K}_i(\mathcal{A})$ is a triangulated subcategory closed under products, hence it is also closed under homotopy limits. Finally it remains to show that $\mathrm{Cot}(E^{\geq -n,\bullet})$ is homotopically injective for all $n \in \mathbb{N}$. But this property holds for bounded below complexes having injective entries (see for example [79, Corollary 10.4.7]), in particular it is true for $\mathrm{Cot}(E^{\geq -n,\bullet})$ too. □

For every complex $X^\bullet \in \mathbf{C}(\mathcal{A})$ having a CE injective resolution $X^\bullet \to E^{\bullet,\bullet}$ we have an obvious map $X^\bullet \to \mathrm{Cot}(E^{\bullet,\bullet})$. Sometimes it happens that this map is a quasi-isomorphism, in which case Lemma 6.1.7 above tells us that it is a homotopically injective resolution. The following lemma shows that this is always the case for bounded below complexes, that is complexes $X^\bullet$ for which $X^n = 0$ for $n << 0$.

Lemma 6.1.8. *Consider a complete abelian category $\mathcal{A}$, which has an injective cogenerator. Let $X^\bullet \in \mathbf{C}(\mathcal{A})$ be a bounded below complex and let $X^\bullet \to E^{\bullet,\bullet}$ be a CE injective resolution. Then $X^\bullet \to \mathrm{Cot}(E^{\bullet,\bullet})$ is a homotopically injective resolution.*

Proof. Without losing the generality, we may suppose that $X^j = 0$ for all $j < 0$, so $E^{i,j} = 0$ for $i < 0$ or $j < 0$. Consider the bicomplex

$$A^{\bullet,\bullet} = 0 \to X^\bullet \to E^{\bullet,0} \to E^{\bullet,1} \to \cdots ,$$

that is the bicomplex whose first column is $X^\bullet$ followed by the columns of $E^{\bullet,\bullet}$ shifted by -1. The sequence of bicomplexes $A^{\bullet,\bullet} \to X^\bullet \to E^{\bullet,\bullet}$ induces a triangle

$$\mathrm{Cot}(A^{\bullet,\bullet}) \to X^\bullet \to \mathrm{Cot}(E^{\bullet,\bullet}) \to \Sigma\mathrm{Cot}(A^{\bullet,\bullet})$$

in $\mathbf{K}(\mathcal{A})$, since $\Sigma\mathrm{Cot}(A^{\bullet,\bullet})$ is the mapping cone of $X^\bullet \to \mathrm{Cot}(E^{\bullet,\bullet})$. Now $A^{\bullet,\bullet}$ is a first quadrant bicomplex (that is $A^{i,j} = 0$ for $i < 0$ or $j < 0$) with acyclic rows. We claim its cototalization is acyclic, and the triangle above proves our lemma.

Because $A^{\bullet,\bullet}$ lies in the first quadrant, it follows that $\mathrm{Cot}(A^{\bullet,\bullet})^n = 0$ for $n < 0$. Fix $n \geq 0$, and let $A^{\leq n+1,\bullet}$ be the truncation of A obtained by deleting the rows in degree $> n+1$, and replacing the $(n+1)$-th row with

$$\cdots \to \mathrm{Z}^{n+1}(A^{i,\bullet}) \to \mathrm{Z}^{n+1}(A^{i+1,\bullet}) \to \cdots .$$

Since, for $0 \leq m \leq n+1$, the computation of $\mathrm{Cot}(A^{\bullet,\bullet})^m$ involves only the rows $A^{i,\bullet}$ with $0 \leq i \leq m$, therefore $\mathrm{Cot}(A^{\bullet,\bullet})^k = \mathrm{Cot}(A^{\leq n,\bullet})^k$, for all $0 \leq k \leq n$. But $A^{\leq n,\bullet}$ is a first quadrant bicomplex with acyclic rows which has only finitely many non–zero rows, therefore we can obtain $\mathrm{Cot}(A^{\leq n,\bullet})$ in finitely many steps by forming triangles whose cones are the rows. This shows that $\mathrm{Cot}(A^{\leq n,\bullet})$ is acyclic, hence $\mathrm{Cot}(A^{\bullet,\bullet})$ is acyclic in degree n. Because n is arbitrary our claim is proved (see also [57, Lemma 2.19]). □

Proposition 6.1.9. *Consider a complete abelian category $\mathcal{A}$, which has an injective cogenerator, such that $\mathbf{D}(\mathcal{A})$ has products. Suppose also that for any complex in $X^\bullet \in \mathbf{C}(\mathcal{A})$ the cototalization of any CE injective resolution $X^\bullet \to E^{\bullet,\bullet}$ provides a homotopically injective resolution $X^\bullet \to \mathrm{Cot}(E^{\bullet,\bullet})$. Then $\mathbf{D}(\mathcal{A})$ has small hom–sets, $\mathbf{D}(\mathcal{A})^o$ is deconstructible and $\mathbf{D}(\mathcal{A})^o$ satisfies Brown representabily.*

Proof. By hypothesis, $X^\bullet \to \mathrm{Cot}(E^{\bullet,\bullet})$ is a homotopically injective resolution, for every $X^\bullet \in \mathbf{K}(\mathcal{A})$. Completing it to a triangle

$$N^\bullet \to X^\bullet \to \mathrm{Cot}(E^{\bullet,\bullet}) \to \Sigma N^\bullet$$

we deduce that $N^\bullet$ is acyclic, that is $\mathbf{K}(\mathcal{A})(N^\bullet, I^\bullet) = 0$ for all $I^\bullet \in \mathbf{K}_i(\mathcal{A})$. By standard arguments concerning Bousfield localizations, see [60, dual of Theorems 9.1.16 and Theorem 9.1.13], we obtain an equivalence of categories

$$\mathbf{K}_i(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}(\mathcal{A}),$$

so $\mathbf{D}(\mathcal{A})$ has small hom–stes.

Note that every complex $X^\bullet$ is isomorphic in $\mathbf{D}(\mathcal{A})$ to $\mathrm{Cot}(E^{\bullet,\bullet})$. Moreover, Lemma 6.1.7 implies $\mathrm{Cot}(E^{\bullet,\bullet}) \cong \underleftarrow{\mathrm{holim}}\mathrm{Cot}(E^{\geq -n,\bullet})$. But, for every $n \in \mathbb{N}$, the kernel of the degree–wise split epimorphism of complexes (see the proof of Lemma 6.1.7) $\mathrm{Cot}(E^{\geq -(n+1),\bullet}) \to \mathrm{Cot}(E^{\geq -n,\bullet})$ is the complex

$$\begin{aligned} 0 \to \mathrm{B}^{-(n+1)}(E^{\bullet,0}) &\to \mathrm{Z}^{-(n+1)}(E^{\bullet,0}) \times \mathrm{B}^{-(n+1)}(E^{\bullet,1}) \\ &\to \mathrm{Z}^{-(n+1)}(E^{\bullet,1}) \times \mathrm{B}^{-(n+1)}(E^{\bullet,2}) \to \cdots \end{aligned}$$

with differentials being represented as matrices whose components are the inclusions $\mathrm{B}^{-(n+1)}(E^{\bullet,j}) \to \mathrm{Z}^{-(n+1)}(E^{\bullet,j})$ and 0 otherwise. Computing the cohomology of this complex we can see that it is quasi-isomorphic, therefore isomorphic in $\mathbf{K}_i(\mathcal{A})$, to the complex:

$$0 \to \mathrm{H}^{-(n+1)}(E^{\bullet,0}) \to \mathrm{H}^{-(n+1)}(E^{\bullet,1}) \to \mathrm{H}^{-(n+1)}(E^{\bullet,2}) \to \cdots,$$

with vanishing differentials. But this last complex is the product of its subcomplexes concentrated in each degree and all entries are injective, hence they are direct summands of a product of copies of Q, where Q is an injective cogenerator of $\mathcal{A}$. Therefore, every object in $\mathbf{K}_i(\mathcal{A})$ is $\mathcal{S}$-cofiltered, for $\mathcal{S} = \{\Sigma^n Q \mid n \in \mathbb{Z}\}$, and Lemma 3.1.3 applies. □

Proof of Theorem 6.1.1. We want to apply Proposition 6.1.9, hence we have to show that, if $\mathbf{D}(\mathcal{A})$ is left–complete, then the cototalization of a CE injective resolution $X^\bullet \to E^{\bullet,\bullet}$ provides a homotopically injective resolution for the complex $X^\bullet \in \mathbf{C}(\mathcal{A})$. This is true for the truncated complexes $X^{\geq -n}$ for all $n \in \mathbb{N}$, by Lemma 6.1.8 above, since $X^{\geq -n} \to E^{\geq -n,\bullet}$ is also a CE injective resolution. Therefore, $X^{\geq -n} \cong \mathrm{Cot}(E^{\geq -n,\bullet})$ in $\mathbf{D}(\mathcal{A})$. Taking homotopy limits and using the hypothesis and Lemma 6.1.7 we obtain:

$$X \cong \underleftarrow{\mathrm{holim}} X^{\geq -n} \cong \underleftarrow{\mathrm{holim}}\mathrm{Cot}(E^{\geq -n,\bullet}) \cong \mathrm{Cot}(E^{\bullet,\bullet})$$

and the proof is complete. □

Remark 6.1.10. For complexes of R-modules, where R is a ring, it is showed in [37] that the cototalization of a CE injective resolution provides a homotopically injective resolution. The technique used there for doing this stresses the so called

Mittag–Leffler condition, which says that limits of inverse towers whose connecting maps are surjective are exact. Amnon Neeman pointed out that Mittag–Leffler condition doesn't work in the more general case of Grothendieck categories, as it may be seen from [70, Corollary 1.6]. Consequently the argument of Keller in [37] may not be used without changes in the case of Grothendieck categories.

In the following Corollary we point out that homotopically injective resolutions exist in $\mathbf{K}(\mathcal{A})$, provided that the abelian category $\mathcal{A}$ satisfies the hypothesis of Theorem 6.1.1.

Corollary 6.1.11. *The following statements hold for a complete abelian category $\mathcal{A}$ possessing an injective cogenerator for which the derived category $\mathbf{D}(\mathcal{A})$ is left–complete:*

(1) Every object in $\mathbf{K}(\mathcal{A})$ has a homotopically injective resolution.

(2) There is an equivalence of categories $\mathbf{K}_i(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}(\mathcal{A})$.

(3) Every additive functor $F : \mathcal{A} \to \mathcal{B}$ to another abelian category $\mathcal{B}$ has a total right derived functor $\mathbf{R}F : \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ (for details see [37, 1.4]).

Proof. As we have already seen the hypotheses of Proposition 6.1.9 are satisfied, hence (1) and (2) hold as it is established in the proof of this Proposition. From here the statement (3) is straightforward. □

Remark 6.1.12. Notice that the conclusions of Corollary 6.1.11 are already known for Grothendieck categories (see [2]). Even if the category $\mathcal{A}$ is not necessary Grothendieck, but it satisfies the hypotheses of Theorem 6.1.1, we can easily prove (1), as follows: Let $X \in \mathbf{K}(\mathcal{A})$¿ Because $\mathcal{T}$ is left–complete X is the homotopy limit in $\mathbf{D}(\mathcal{A})$ of an inverse tower

$$X^{\geq 0} \leftarrow X^{\geq -1} \leftarrow X^{\geq -2} \leftarrow \cdots .$$

whose terms are bounded below complexes. Replacing every term of this tower with a homotopically injective resolution, which exists by Lemma 6.1.8, we obtain an inverse tower

$$E^{\geq 0} \leftarrow E^{\geq -1} \leftarrow E^{\geq -2} \leftarrow \cdots$$

which is quasi-isomorphic to the first one. Then we get a quasi-isomorphism $X \to \underleftarrow{\mathrm{holim}}_{n\leq 0} E^{\geq n}$, providing a homotopically injective resolution for X. Now (2) and (3) are straightforward. The problem with this shortest argument is that we have less control over the complexes $E^{\geq n}$, in particular we do not know that the kernel the map $E^{\geq n} \leftarrow E^{\geq n-1}$ has vanishing differentials, hencefort it is not clear if Brown representability for $\mathbf{D}(\mathcal{A})^o$ can be deduced from it.

6.2 Cogenerators in triangulated categories

An *ideal* in an additive category $\mathcal{A}$ is a collection of morphisms which is closed under addition and composition with arbitrary morphisms in $\mathcal{A}$. For $s \in \mathbb{N}^*$, the *s-th power* of an ideal $\mathcal{I}$ denoted $\mathcal{I}^s$ is the ideal generated (that is the closure under addition) of the set

$$\{f \mid \text{there are } f_1, \dots, f_s \in \mathcal{I} \text{ such that } f = f_1 \cdots f_s\}.$$

If $\mathcal{I}$ and $\mathcal{J}$ are ideals, then to show that $\mathcal{I}^s \subseteq \mathcal{J}$ it is obviously enough to show that the generators $f = f_1 \cdots f_s$ lie in $\mathcal{J}$. From now on $\mathcal{S} \subseteq \mathcal{T}$ is a Σ-stable set. We call $\mathcal{S}$-*(co)phantom* a map $\phi : X \to Y$ with the property $\mathcal{T}(\mathcal{S}, \phi) = 0$ (respectively $\mathcal{T}(\phi, \mathcal{S}) = 0$). (The notations $\mathcal{T}(\mathcal{S}, \phi) = 0$ and $\mathcal{T}(\phi, S) = 0$ mean $\mathcal{T}(S, \phi) = 0$, respectively $\mathcal{T}(\phi, S) = 0$, for all $S \in \mathcal{S}$.) Observe that $\phi : X \to Y$ is a phantom if and only if for every map $S \to X$ with $S \in \mathcal{S}$, the composite map $S \to X \xrightarrow{\phi} Y$ vanishes, and dual for a cophantom. We denote

$$\Phi(\mathcal{S}) = \{\phi \mid \phi \text{ is an } \mathcal{S}\text{-phantom}\} \text{ and } \Psi(\mathcal{S}) = \{\psi \mid \phi \text{ is an } \mathcal{S}\text{-cophantom}\}.$$

Clearly $\Phi(\mathcal{S})$ and $\Psi(\mathcal{S})$ are Σ-stable ideals in $\mathcal{T}$, that is they are also closed under Σ and Σ^{-1}. The ideals defined above depend on the ambient category $\mathcal{T}$. If we want to emphasize this dependence we will write $\Phi_{\mathcal{T}}(\mathcal{S})$, respectively $\Psi_{\mathcal{T}}(\mathcal{S})$.

Lemma 6.2.1. *If $\mathcal{C}$ is a set of objects in $\mathcal{T}$ then every $Y \in \mathcal{T}$ has a* $\mathrm{Prod}(\mathcal{C})$*–preenvelope $Y \to Z$. Moreover if $\mathcal{C}$ is also Σ-stable, then this preenvelope fits in a triangle $X \overset{\psi}{\to} Y \to Z \to \Sigma X$, with $\psi \in \Psi(\mathcal{C})$.*

Proof. The argument is standard: Let $Z = \prod_{C\in\mathcal{C},\alpha:Y\to C} C$ and $Y \to Z$ the unique map making commutative the diagram:

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr, "\alpha"'] & & Z \arrow[dl, "p_{C,\alpha}"] \\ & C & \end{tikzcd}$$

where $p_{C,\alpha}$ is the canonical projection for all $C \in \mathcal{C}$ and all $\alpha : Y \to C$. For a Σ-stable set $\mathcal{C}$, we complete this map to a triangle

$$X \overset{\psi}{\to} Y \to Z \to \Sigma X.$$

It may be immediately seen that the condition to be a $\mathrm{Prod}(\mathcal{C})$–preenvelope is equivalent to $\psi \in \Psi(\mathcal{C})$. $\square$

Lemma 6.2.2. *Assume that $\mathcal{C} \subseteq \mathcal{T}$ and $\mathcal{G} \subseteq \mathcal{T}$ are two Σ-stable sets, such that there is $s \in \mathbb{N}^*$ with the property $\Psi(\mathcal{C})^s \subseteq \Phi(\mathcal{G})$. Then every $Y \in \mathcal{T}$ fits in a triangle $X \overset{\phi}{\to} Y \to Z \to \Sigma X$, with $Z \in \mathrm{Prod}_s(\mathcal{C})$ and $\phi \in \Phi(\mathcal{G})$.*

Proof. We begin with an inductive construction. First denote $X_1 = Y$, and if X_k is already constructed, $k \in \mathbb{N}^*$, then use Lemma 6.2.1 to construct the triangle

$$X_{k+1} \stackrel{\psi_k}{\to} X_k \to P_k \to \Sigma X_{k+1},$$

where $X_k \to P_k$ is a $\operatorname{Prod}(\mathcal{C})$–preenvelope of X_k and $\psi_k \in \Psi(\mathcal{C})$. Define also $\psi^1 = 1_Y : X_1 \to Y$ and $\psi^{k+1} = \psi^k \psi_k : X_{k+1} \to Y$. Next complete them to triangles $X_k \stackrel{\psi^k}{\to} Y \to Z_k \to \Sigma X_k$, for all $k \in \mathbb{N}^*$. The octahedral axiom allows us to construct the commutative diagram whose rows and columns are triangles:

$$\begin{tikzcd}
 & \Sigma^{-1}Z_k \arrow[r, equal] \arrow[d] & \Sigma^{-1}Z_k \arrow[d] & \\
X_{k+1} \arrow[r, "\psi_k"] \arrow[d, equal] & X_k \arrow[r] \arrow[d, "\psi^k"] & P_k \arrow[r] \arrow[d] & \Sigma X_{k+1} \arrow[d, equal] \\
X_{k+1} \arrow[r, "\psi^{k+1}"'] & Y \arrow[r] \arrow[d] & Z_{k+1} \arrow[r] \arrow[d] & \Sigma X_{k+1} \\
 & Z_k \arrow[r, equal] & Z_k &
\end{tikzcd}.$$

We have $Z_1 = 0$, $Z_2 \cong P_1 \in \operatorname{Prod}(\mathcal{C})$ and the triangle in the second column of the above diagram allows us to complete the induction step in order to show that $Z_{k+1} \in \operatorname{Prod}_k(\mathcal{C})$. Clearly we also have $\psi^{k+1} \in \Psi(\mathcal{C})^k$, thus the desired triangle is $X_{s+1} \xrightarrow{\psi^{s+1}} Y \longrightarrow Z_{s+1} \longrightarrow \Sigma X_{s+1}$. $\square$

Lemma 6.2.3. *Assume that $\mathcal{C} \subseteq \mathcal{T}$ and $\mathcal{G} \subseteq \mathcal{T}$ are two Σ-stable sets, such that there is $s \in \mathbb{N}^*$ with the property $\Psi(\mathcal{C})^s \subseteq \Phi(\mathcal{G})$. Then every map $Y \to Z$ in $\mathcal{T}$ with $Z \in \operatorname{Prod}_n(\mathcal{C})$ factors as $Y \to Z' \to Z$, where $Z' \in \operatorname{Prod}_{n+s}(\mathcal{C})$ and the induced maps*

$$\mathcal{T}(G, Y) \to \mathcal{T}(G, Z) \text{ and } \mathcal{T}(G, Z') \to \mathcal{T}(G, Z)$$

have the same image, for all $G \in \mathcal{G}$.

Proof. Complete $Y \to Z$ to a triangle $Y \to Z \to Y' \to \Sigma Y$ and let

$$X \stackrel{\phi}{\to} Y' \to Z'' \to \Sigma X,$$

with $\phi \in \Phi(\mathcal{G})$ and $Z'' \in \operatorname{Prod}_s(\mathcal{C})$ the triangle whose existence is proved in Lemma 6.2.2. Complete the composed map $Z \to Y' \to Z''$ to a triangle

$$Z' \to Z \to Z'' \to \Sigma Z'.$$

It is clear that $Z' \in \operatorname{Prod}_{n+s}(\mathcal{C})$. We can construct the commutative diagram:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & Y' \arrow[r] \arrow[d] & \Sigma Y \arrow[d] \\
Z' \arrow[r] & Z \arrow[r] & Z'' \arrow[r] & \Sigma Z'
\end{tikzcd}$$

by completing the middle square with $Y \to Z'$ in order to obtain a morphism of triangles. Applying the functor $\mathcal{T}(G,-)$ with an arbitrary $G \in \mathcal{G}$ we get a commutative diagram with exact rows:

$$\begin{tikzcd} \mathcal{T}(G,Y) \arrow[r] \arrow[d] & \mathcal{T}(G,Z) \arrow[r] \arrow[d, equal] & \mathcal{T}(G,Y') \arrow[d] \\ \mathcal{T}(G,Z') \arrow[r] & \mathcal{T}(G,Z) \arrow[r] & \mathcal{T}(G,Z'') \end{tikzcd} .$$

Since $\phi \in \Phi(\mathcal{G})$ we deduce $\mathcal{T}(G,Y') \to \mathcal{T}(G,Z'')$ is injective, so the kernels of the two right hand parallel arrows are the same. But these kernels coincide to the images of the two left hand parallel arrows. $\square$

A diagram of triangulated categories and functors of the form $\mathcal{L} \stackrel{I}{\to} \mathcal{T} \stackrel{Q}{\to} \mathcal{U}$ is called a *localization sequence* if I is fully faithful and has a right adjoint, $\operatorname{Ker} Q = \operatorname{Im} I$ and Q has a right adjoint too. By formal non-sense (see [60, Theorem 9.1.16]) this right adjoint Q_ρ of Q has also to be fully faithful and makes $\mathcal{U}$ equivalent to the category $(\operatorname{Im} I)^\perp$.

Theorem 6.2.4. *Let $\mathcal{G} \subseteq \mathcal{T}$ be a Σ-stable set and denote $\mathcal{U} = \left(\mathcal{G}^\perp\right)^\perp$. Suppose that there is a Σ-stable set $\mathcal{C} \subseteq \mathcal{U}$ and an integer $s \in \mathbb{N}^*$ such that $\Psi(\mathcal{C})^s \subseteq \Phi(\mathcal{G})$. Then $\mathcal{U} = \operatorname{Coloc}(\mathcal{C})$, there is a localization sequence $\mathcal{G}^\perp \to \mathcal{T} \to \mathcal{U}$ and $\mathcal{U}^o$ satisfies Brown representability.*

Proof. Note first that, by its very construction, $\mathcal{U}$ is triangulated and closed under products in $\mathcal{T}$. Fix $Y \in \mathcal{T}$. Construct as in Lemma 6.2.2 a triangle $X_1 \stackrel{\phi_1}{\to} Y \to Z_1 \to \Sigma X_1$, with $Z_1 \in \operatorname{Prod}_s(\mathcal{C})$ and $\phi_1 \in \Phi(\mathcal{G})$. We use Lemma 6.2.3 in order to inductively construct maps $Y \to Z_n$, with $Z_n \in \operatorname{Prod}_{sn}(\mathcal{C})$, $n \in \mathbb{N}^*$, such that every $Y \to Z_n$ factors as $Y \to Z_{n+1} \to Z_n$, with the abelian group homomorphisms $\mathcal{T}(G,Y) \to \mathcal{T}(G,Z_n)$ and $\mathcal{T}(G,Z_{n+1}) \to \mathcal{T}(G,Z_n)$ having the same image, for all $G \in \mathcal{G}$. From now on the argument runs as in the proof of [64, Theorem 4.7]. We recall it here for the reader's convenience: Let $Z = \underleftarrow{\operatorname{holim}}\, Z_n$. We have constructed the commutative diagram in $\mathcal{T}$:

$$\begin{tikzcd} Y \arrow[r, equal] \arrow[d] & Y \arrow[r, equal] \arrow[d] & Y \arrow[r, equal] \arrow[d] & \cdots \\ Z_1 & Z_2 \arrow[l] & Z_3 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

inducing a map $Y \to Z$. Fix $G \in \mathcal{G}$. Applying the functor $\mathcal{T}(G,-)$ to above diagram in $\mathcal{T}$ we get a commutative diagram of abelian groups

$$\begin{tikzcd} \mathcal{T}(G,Y) \arrow[r, equal] \arrow[d] & \mathcal{T}(G,Y) \arrow[r, equal] \arrow[d] & \mathcal{T}(G,Y) \arrow[r, equal] \arrow[d] & \cdots \\ \mathcal{T}(G,Z_1) & \mathcal{T}(G,Z_2) \arrow[l] & \mathcal{T}(G,Z_3) \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

with the first (hence all) vertical map(s) injective, and the images of both maps ending in each $\mathcal{T}(G, Z_n)$, $n \in \mathbb{N}^*$, coincide. This shows that the tower below is the direct sum of the above one and a tower with vanishing connecting maps, hence $\mathcal{T}(G, Y) \cong \lim \mathcal{T}(G, Z_n)$ canonically. Moreover the inverse limit of the second row has to be exact, thus we obtain a short exact sequence:

$$0 \to \lim \mathcal{T}(G, Z_n) \longrightarrow \prod_{n\in\mathbb{N}^*} \mathcal{T}(G, Z_n) \stackrel{1-shift}{\longrightarrow} \prod_{n\in\mathbb{N}^*} \mathcal{T}(G, Z_n) \to 0.$$

Comparing this sequence with the one obtained by applying $\mathcal{T}(G, -)$ to the triangle

$$Z \longrightarrow \prod_{n\in\mathbb{N}^*} Z_n \stackrel{1-shift}{\longrightarrow} \prod_{n\in\mathbb{N}^*} Z_n \longrightarrow \Sigma Z$$

we deduce $\lim \mathcal{T}(G, Z_n) \cong \mathcal{T}(G, Z)$. Therefore the map $Y \to Z$ constructed above induces an isomorphism $\mathcal{T}(G, Y) \stackrel{\cong}{\longrightarrow} \mathcal{T}(G, Z)$. Complete $Y \to Z$ to a triangle

$$X \to Y \to Z \to \Sigma X.$$

Since G was arbitrary, we deduce $X \in \mathcal{G}^\perp$ and obviously $Z \in \mathcal{U}$. Therefore the triangle above corroborated with [60, Theorem 9.1.13] proves that there is a localization sequence $\mathcal{G}^\perp \to \mathcal{T} \to \mathcal{U}$. Finally supposing $Y \in \mathcal{U}$ this forces $X \in \mathcal{U}$, because $\mathcal{U}$ is triangulated. Since we have also $X \in \mathcal{G}^\perp$ we infer $X = 0$, thus $Y \cong Z = \underleftarrow{\operatorname{holim}}\, Z_n \in \operatorname{Coloc}(\mathcal{C})$, hence $\mathcal{U} = \operatorname{Coloc}(\mathcal{C})$ is $\mathcal{S}$–cofiltered and all we need is to apply Theorem 3.1.3. □

Corollary 6.2.5. *Assume that $\mathcal{C} \subseteq \mathcal{T}$ and $\mathcal{G} \subseteq \mathcal{T}$ are two Σ-stable sets, such that there is $s \in \mathbb{N}^*$ with the property $\Psi(\mathcal{C})^s \subseteq \Phi(\mathcal{G})$, and assume also that $\mathcal{G}$ generates $\mathcal{T}$. Then $\mathcal{T} = \operatorname{Coloc}(\mathcal{C})$ and $\mathcal{T}^o$ satisfies Brown representability.*

Proof. The hypothesis $\mathcal{G}$ generates $\mathcal{T}$ means $\mathcal{G}^\perp = \{0\}$. Thus one applies Theorem 6.2.4 with $\mathcal{U} = \left(\mathcal{G}^\perp\right)^\perp = \mathcal{T}$. □

Corollary 6.2.6. *Let $\mathcal{G} \subseteq \mathcal{T}$ be a Σ-stable set and denote $\mathcal{U} = \left(\mathcal{G}^\perp\right)^\perp$. Suppose that there is a Σ-stable set $\mathcal{C} \subseteq \mathcal{U}$ and an integer $s \in \mathbb{N}^*$ such that $\Psi(\mathcal{C})^s \subseteq \Phi(\mathcal{G})$. Suppose in addition that $\mathcal{T}$ has coproducts and there is a localization sequence $\operatorname{Loc}(\mathcal{G}) \to \mathcal{T} \to \mathcal{G}^\perp$. Then $\operatorname{Loc}(\mathcal{G})$ is equivalent to $\mathcal{U}$ and, consequently, $\operatorname{Loc}(\mathcal{G})^o$ satisfies Brown representability. In particular, a localization sequence as above exists, provided that objects in $\mathcal{G}$ are α-compact, for a regular cardinal α.*

Proof. First apply Theorem 6.2.4 in order to obtain a localization sequence $\mathcal{G}^\perp \to \mathcal{T} \to \mathcal{U}$. Together with the localization sequence whose existence is supposed in the hypothesis, this shows that both categories $\operatorname{Loc}(\mathcal{G})$ and $\mathcal{U}$ are equivalent to the Verdier quotient $\mathcal{T}/\mathcal{G}^\perp$, hence they are equivalent to each other. Finally provided that objects in $\mathcal{G}$ are α-compact, we know by [60, Theorem 8.4.2] that $\operatorname{Loc}(\mathcal{G})$ satisfies Brown representability. Consequently the inclusion functor $\operatorname{Loc}(\mathcal{G}) \to \mathcal{T}$ which preserves coproducts must have a right adjoint and a localization sequence $\operatorname{Loc}(\mathcal{G}) \to \mathcal{T} \to \mathcal{G}^\perp$ exists. □

In the end of this section let observe that the general version of Brown representability for covariant functors proved in [43] is a consequence of our criterion. In order to do that, let $\mathcal{T}$ be a triangulated category with products and coproducts. Recall from [43, Definition 2] that a *set of symmetric generators* for $\mathcal{T}$ is a set $\mathcal{G} \subseteq \mathcal{T}$ which generates $\mathcal{T}$ such that and there is another set $\mathcal{C} \subseteq \mathcal{T}$ with the property that for every map $X \to Y$ in $\mathcal{T}$ the induced map $\mathcal{T}(G, X) \to \mathcal{T}(G, Y)$ is surjective for all $G \in \mathcal{G}$ if and only if the induced map $\mathcal{T}(Y, C) \to \mathcal{T}(X, C)$ is injective for all $C \in \mathcal{C}$. Completing the map $X \to Y$ to a triangle it is easy to see that the last condition is equivalent to the fact $\Phi(\mathcal{G}) = \Psi(\mathcal{C})$. Remark also that without losing the generality, we may suppose the sets $\mathcal{G}$ and $\mathcal{C}$ to be Σ-closed. Applying Corollary 6.2.5 we obtain:

Corollary 6.2.7. *[43, Theorem B] If $\mathcal{T}$ has products, coproducts and a set of symmetric generators, then $\mathcal{T}^o$ satisfies Brown representability.*

Remark also that hypotheses in [43, Theorem B] are general enough to include the case of compactly generated categories.

6.3 The dual of Brown representability for homotoy category of projectives

Recall that a ring with several objects is a small preadditive category R, and an R-module is a functor $R^o \to \mathcal{A}b$. (Our modules are right modules by default.) Clearly if the category R has exactly one object, then it is nothing else than an ordinary ring with unit, and modules are abelian groups endowed with a multiplication with scalars from R. In the sequel, the category $\mathcal{A}$ will be often an additive exact (that is closed under extensions) subcategory of the category $\mathrm{Mod}(R)$ of modules over a ring with several objects R. For example, $\mathcal{A}$ may be $\mathrm{Flat}(R)$ of $\mathrm{Proj}(R)$ the full subcategories of all flat, respectively projective modules. Another source of examples is the subcategory of projective complexes over a module category R, that is $\mathrm{Proj}(\mathbf{C}(R))$. Note then that if $\mathcal{A} \subseteq \mathrm{Mod}(R)$ or $\mathcal{A} \subseteq \mathbf{C}(\mathrm{Mod}(R))$ an additive exact category as above, then $\mathbf{K}(\mathcal{A})$ is triangulated subcategory of $\mathbf{K}(\mathrm{Mod}(R))$ respectively $\mathbf{K}(\mathbf{C}(\mathrm{Mod}(R)))$.

In the proof of the next Lemma and Theorem we will use several results in [62], [64] and [63]; note that even they are stated for rings with one, they don't make use of the existence of the unit, and the same arguments can be used for rings with several objects.

Lemma 6.3.1. *The category $\mathbf{K}(\mathrm{Proj}(R))$ has products.*

Proof. From [62, Theorem 1.1] we learned that $\mathbf{K}(\mathrm{Proj}(R))$ is well–generated, hence it satisfies Brown representability. The existence of products follows by Theorem 1.4.4. □

Theorem 6.3.2. *If R is a ring with several objects, then $\mathbf{K}(\mathrm{Proj}(R))^o$ satisfies Brown representability.*

Proof. We want to apply Corollary 6.2.6, therefore we have to verify that the assumptions made there are fulfilled. First the category $\mathbf{K}(\mathrm{Proj}(R))$ has products, by Lemma 6.3.1 above.

The ambient category is $\mathcal{T} = \mathbf{K}(\mathrm{Flat}(R))$. We know by [62, Theorem 5.9] that is full subcategory which contains a set of representatives, up to homotopy equivalence, for bounded below complexes with finitely generated projective entries is a a set of $\aleph_1$-compact generators for $\mathbf{K}(\mathrm{Proj}(R))$. This subcategory is considered in [62, Construction 4.3]. Let $\mathcal{G}$ be the closure under suspensions and desuspensions of this generating set. Clearly $\mathcal{G}$ also generates $\mathbf{K}(\mathrm{Proj}(R))$ and we have $\mathbf{K}(\mathrm{Proj}(R)) = \mathrm{Loc}(\mathcal{G})$. Denote

$$\mathcal{U} = \left(\mathcal{G}^{\perp}\right)^{\perp} = \left(\mathbf{K}(\mathrm{Proj}(R))^{\perp}\right)^{\perp}.$$

According to [64, Definition 1.3], a map $X \to Y$ of complexes of R-modules is called a *tensor phantom map*, if the induced map $X \otimes_R I \to Y \otimes_R I$ vanishes in cohomology for every test–complex I.

Following [64, Definition 1.1], we call *test–complex* a bounded below complex I of injective left R-modules satisfying the additional properties that $\mathrm{H}^n(I) = 0$ for all but finitely many $n \in \mathbb{Z}$ and for those n for which $\mathrm{H}^n(I) \neq 0$, this module is isomorphic to subquotient of a finitely generated projective module. According to [64, Remark 1.2] there is only a set of test complexes up to homotopy equivalence. Note that [63, Theorem 3.2] states that the inclusion functor $\mathbf{K}(\mathrm{Flat}(R)) \to \mathbf{K}(\mathrm{Mod}(R))$ has a right adjoint, which we denote here by

$$J : \mathbf{K}(\mathrm{Mod}(R)) \to \mathbf{K}(\mathrm{Flat}(R)).$$

Define $\mathcal{C}$ to be the closure under Σ and Σ^{-1} of the the full subcategory of $\mathcal{T} = \mathbf{K}(\mathrm{Flat}(R))$ which contains exactly the objects of the form $J(\mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z}))$ where I runs over a set of representatives up to homotopy equivalence of all test–complexes. Then [64, Lemma 2.6] implies $\mathcal{C} \subseteq \mathcal{U}$. According to [64, Lemma 2.8], we have

$$\Psi(\mathcal{C}) = \{\psi \mid \psi \text{ is a tensor phantom map}\}.$$

Moreover $\Psi(\mathcal{C})^2 \subseteq \Phi(\mathcal{G})$, as [64, Lemma 1.9] states. Therefore Corollary 6.2.6 applies, hence $\mathbf{K}(\mathrm{Proj}(R))^o = \mathrm{Loc}(\mathcal{G})^o$ satisfies Brown representability. □

The following Corollary gives an argument for the existence of the left adjoint of the inclusion functor $\mathcal{U} \to \mathbf{K}(\mathrm{Flat}(R))$ is a consequence of Theorem 6.3.2. By now there are several proof of this fact (see [64]), but the new one is deduced more conceptually as a consequence of Brown representability.

Corollary 6.3.3. *With the notations made in the proof of Theorem 6.3.2 The inclusion functor $\mathcal{U} \to \mathbf{K}(\mathrm{Flat}(R))$ has a left adjoint.*

Proof. By Corollary 6.2.6, $\mathcal{U}$ is equivalent to $\mathbf{K}(\mathrm{Proj}(R))$, so it satisfies Brown representability, and the conclusion follows by 1.4.3. □

Pure–projective modules can be seen as projective modules over a suitable ring with several objects. This idea leads to:

Corollary 6.3.4. *If R is a ring, then the dual of the homotopy category of pure–projective modules satisfies Brown representability.*

Proof. Let R be a ring and denote by $\mathrm{Pproj}(R)$ the category of pure projective R-modules. Denote by $A = \mathrm{mod}(R)$ the full subcategory of $\mathrm{Mod}(R)$ which consists of all finitely presented modules, and regard it as a ring with several objects. It is well known that the functor

$$\mathrm{Mod}(R) \to \mathrm{Mod}(A),\ X \mapsto \mathrm{Hom}_R(-, X)|_A$$

is an embedding and restrict to an equivalence $\mathrm{Pproj}(R) \stackrel{\sim}{\to} \mathrm{Proj}(A)$. Now apply Theorem 6.3.2. □

Recall that a ring is pure semisimple if every R-module is pure projective. Therefore from Corollary 6.3.4 we can derive a new proof for an already known result (see Theorem 5.2.10 and Remark 5.2.11):

Corollary 6.3.5. *If the ring R is pure semisimple then $\mathbf{K}(\mathrm{Mod}(R))^o$ satisfies Brown representability.*

In the following we want to study Brown representability for the dual of the homotopy category of projective representations over a quiver. In order to perform this, we note first that coreflective subcategories in the sense of [1] inherits deconstructibity:

Proposition 6.3.6. *Let $U : \mathcal{L} \to \mathcal{T}$ be a fully faithful functor which has a right adjoint $F : \mathcal{T} \to \mathcal{L}$. If $\mathcal{T}^o$ is deconstructible, then $\mathcal{L}^o$ is so and, consequently, $\mathcal{L}^o$ satisfies Brown representability.*

Proof. By hypothesis there is a set $\mathcal{C}$ such that $\mathcal{T}$ is $\mathcal{C}$–cofiltered. We shall show that $\mathcal{L}$ is $F(\mathcal{C})$-filtered, and we are done. Let $X \in \mathcal{L}$. Then for $U(X) \in \mathcal{T}$ there is an inverse tower

$$0 = X_0 \leftarrow X_1 \leftarrow \cdots$$

such that $U(X) \cong \underleftarrow{\mathrm{holim}}_{n\in\mathbb{N}} X_n$ and in the triangle $X_{n+1} \to X_n \to P_n \to \Sigma X_{n+1}$ we have $P_n \in \mathrm{Prod}(\mathcal{C})$. Applying the product preserving triangulated functor F we obtain $FU(X) \cong \underleftarrow{\mathrm{holim}}_{n\in\mathbb{N}} F(X_n)$ and in the triangle $F(X_{n+1}) \to F(X_n) \to F(P_n) \to \Sigma F(X_{n+1})$ we have $F(P_n) \in \mathrm{Prod}(F(\mathcal{C}))$. Finally it remains only to note that $X \cong FU(X)$ since U is supposed to be fully faithful. □

Recall that a *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$ where Q_0 and Q_1 are disjoint sets whose elements are called *vertices*, respectively *arrows* of Q, and $s, t : Q_1 \to Q_0$ are two maps. If for $m \in Q_1$ we have $i = s(m)$ and $j = t(m)$ then we call the vertices i and j the source, respectively the target of the arrow m. We write $m : i \to j$ to indicate this fact. Two arrows $m, m' \in Q_1$ are composable if $s(m') = t(m)$. If this is the case we denote by $m'm$ the composition and we set $s(m'm) = s(m)$ and $t(m'm) = t(m')$. We have just obtained a path of length 2. Generalizing this, a *path* in the quiver Q is a finite sequence of composable maps; the number of the maps occuring in a path is called the *length* of the path. Vertices are seen as paths of length 0, or trivial paths. A relation in a quiver is obtained as following: Consider $\{(\gamma_i, \delta_i) \mid i \in I\}$ an arbitrary set of pair of paths such that $s(\gamma_i) = s(\delta_i)$ and $t(\gamma_i) = t(\delta_i)$ for all $i \in I$. We put $\gamma_i \sim \delta_i$, and whenever σ and τ are paths such that the compositions make sense we have $\sigma\gamma_i \sim \sigma\delta_i$ and $\gamma_i\tau \sim \delta_i\tau$. Moreover for any path γ we set $\gamma \sim \gamma$. It is easy to see that $\sim$ will be an equivalence relation on the set of all paths in Q. It is clear that every quiver may be seen as a quiver with relations, since equality is the poorest equivalence relation. Henceforth by a quiver we will always mean a quiver with relations. A representation X of the quiver Q in $\mathrm{Mod}(R)$ is an assignment to each vertex $i \in Q_0$ an R-module $X(i)$ and to each arrow $m : i \to j$ in Q_1 an R-linear map $X(m) : X(i) \to X(j)$, such that equivalent paths lead to equal linear maps. Morphisms of representations $f : X \to Y$ are collections of R-linear maps $f = (f_i : X(i) \to Y(i))_{i \in Q_0}$, with the property $f_j X(m) = Y(m) f_i$ for any arrow $m : i \to j$ in Q_1. We obtain a category, namely the category of representations of Q in $\mathrm{Mod}(R)$ denoted $\mathrm{Mod}(R, Q)$. Further, let A be the free R-module with the basis B the set of all paths in Q modulo the equivalence relation above, that is

$$A = R^{(B)} = \bigoplus_{b \in B} bR,$$

where $bR = \{br \mid r \in R\}$ is a copy of R as right R-module. For two elements $b = [\gamma]$ and $b' = [\gamma']$ in B we define the product $b'b = [\gamma'\gamma]$ if the paths γ and γ' are composable, and $b'b = 0$ otherwise. Declaring that elements in R commute with all elements of the base, the product extends by distributivity to all elements in A, making A into an R-algebra, the so called *path algebra* of Q over R. The trivial paths lead to a family of orthogonal idempotents $e = e_i \in A$, with $i \in Q_0$. If Q_0 is finite, then $\sum_{i \in Q_0} e_i$ is unit in A, otherwise we may add an extra element 1 to the basis B which acts as unit, that is $1b = b1 = b$ for any path b. A slightly different (but equivalent) approach of the matter concerning quivers, may be found in [48, Chapter II,§1] (quivers are called there diagram schemes). The categories $\mathrm{Mod}(R, Q)$ and $\mathrm{Mod}(A)$ are linked by two functors, namely $U : \mathrm{Mod}(R, Q) \to \mathrm{Mod}(A)$, given by $U(X) = \bigoplus_{i \in Q_0} X(i)$, and $F : \mathrm{Mod}(A) \to \mathrm{Mod}(R, Q)$, $F(M)(i) = Me_i$.

Lemma 6.3.7. *With the above notations the following statements hold:*

a) The functor F is the right adjoint of U.

b) U is fully faithful.

c) Both F and U preserve projective objects.

Proof. a) Let $X \in \mathrm{Mod}(R, Q)$ and $M \in \mathrm{Mod}(A)$. If $f : \bigoplus_{i \in Q_0} X(i) \to M$ in an A-linear map, then for every $x \in X(i)$ we have $f(x) = f(xe_i) = f(x)e_i \in Me_i$, showing that $f = (f_i : X(i) \to Me_i)_{i \in Q_0}$. It is not hard to see that the R-linear maps f_i have to commute with the maps induced by every $i \to j$ in Q_1, so f is a map of representations $X \to F(M)$. Conversely if $f_i : X(i) \to Me_i)_{i \in I}$ is a map of representations, then the family of maps $(X(i) \to Me_i \to M)_{i \in I}$ induce a unique R-linear map $f : \bigoplus_{i\in I} X_i \to M$. Moreover if $a \in A$ and $x \in \bigoplus_{i\in I} X_i$ then both are written as finite sums $a = \sum_{b\in B} ba_b$ and $x = \sum_{i\in I} x_i$ with $a_b \in R$ and $x_i \in X(i)$ (almost all zero). By distributivity $f(xa) = f(x)a$, hence f is A-linear. This proves the adjunction between F and U.

b) Let X be a representation of Q. We have

$$F(U(X)) = F\left(\bigoplus_{i\in I} X(i)\right) = (X(i))_{i\in I} = X$$

therefore the functor U is fully faithful.

c) First observe that for $M \in \mathrm{Mod}(A)$, we have $F(M)(i) = Me_i$, $i \in I$, and Me_i is a direct summand of M, hence F is an exact functor. This implies that its left adjoint U preserves projective objects. Moreover F preserves coproducts. So for showing that F preserves projective objects it is enough to show that $F(A)$ is projective in $\mathrm{Mod}(R, Q)$. In order to prove this we will determine better the projective objects in $\mathrm{Mod}(R, Q)$. View Q as a small category with object set Q_0 and maps equivalence classes of paths in Q. By [48, Chapter II, §12], $\mathrm{Mod}(R, Q)$ is equivalent to the category of functors from this small category to $\mathrm{Mod}(R)$, consequently according to [48, Chapter VI, Theorem 4.3], projectives in $\mathrm{Mod}(R, Q)$ are exactly

$$\mathrm{Proj}(R, Q) = \mathrm{Add}(\{S_i(P) \mid i \in Q_0 \text{ and } P \in \mathrm{Proj}(R)\}),$$

where $S_i : \mathrm{Mod}(R) \to \mathrm{Mod}(R, Q)$ are functors defined by

$$S_i(V) = \bigoplus_{i/Q_0} V \text{ for all } V \in \mathrm{Mod}(R),$$

$i/Q_0 = \{([\gamma], j) \mid j \in Q_0 \text{ and } \gamma : i \to j \text{ is a path}\}$, and by Add we understand the closure under direct sums and direct summands (that is, the dual of Prod). Now, since $F(A)(i) = Ae_i$, for all $i \in Q_0$, we deduce $F(A) \in \mathrm{Proj}(R, Q)$. $\square$

Theorem 6.3.8. *Let Q be a quiver and denote by* $\mathrm{Proj}(R, Q)$ *the category of projective objects in the category* $\mathrm{Mod}(R, Q)$. *Then* $\mathbf{K}(\mathrm{Proj}(R, Q))^o$ *satisfies Brown representability.*

Proof. Consider the path algebra A of the quiver Q and the pair of adjoint functors $U : \mathrm{Mod}(R, \mathcal{I}) \leftrightarrows \mathrm{Mod}(A) : F$ defined above. By Lemma 6.3.7 we obtained a pair of adjoint functors between $\mathrm{Proj}(R, Q)$ and $\mathrm{Proj}(A)$, which extends to a pair of triangulated adjoint functors (denoted with the same symbols)

$$U : \mathbf{K}(\mathrm{Proj}(R, Q)) \leftrightarrows \mathbf{K}(\mathrm{Proj}(A)) : F\ .$$

In addition we know that the initial U is fully faithful, so the same is true for the extended functor. By Theorem 6.3.2, $\mathbf{K}(\mathrm{Proj}(A))$ is deconstructible, hence Proposition 6.3.6 applies. $\square$

Example 6.3.9. If in Theorem 6.3.8 we put Q to be the following quiver

$$\cdots \to i-1 \overset{\partial^{i-1}}{\to} i \overset{\partial^{i}}{\to} i+1 \to \cdots, \ (i \in \mathbb{Z}),$$

with relations $\partial^i \partial^{i-1} = 0$, then a representation of this quiver is, obviously, a complex over R. Therefore $\mathrm{Mod}(R, Q) = \mathbf{C}(\mathrm{Mod}(R))$, and we obtain a proof for the fact that the dual of the homotopy category of projective complexes of R-modules satisfies Brown representability.

Appendix A

Further research

We start by recalling an important notion used in the study of triangulated categories: A *t-structure* in a triangulated category in $\mathcal{T}$ (see [7, Definition 1.3.1]) is a pair of full subcategories $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$, such that the following conditions are satisfied, where we use the notations $\mathcal{T}^{\geq n} = \Sigma^{-n}\mathcal{T}^{\geq 0}$ and $\mathcal{T}^{\leq n} = \Sigma^{-n}\mathcal{T}^{\leq 0}$:

(T1) $\mathcal{T}(X, Y) = 0$ for all $X \in \mathcal{T}^{\leq 0}$ and all $Y \in \mathcal{T}^{\geq 1}$.

(T2) $\mathcal{T}^{\leq 0} \subseteq \mathcal{T}^{\leq 1}$ and $\mathcal{T}^{\geq 1} \subseteq \mathcal{T}^{\geq 0}$

(T3) For all $X \in \mathcal{T}$ there is a triangle

$$X^{\leq 0} \to X \to X^{\geq 1} \to \Sigma X^{\leq 0}$$

in $\mathcal{T}$, with $X^{\leq 0} \in \mathcal{T}^{\leq 0}$ and $X^{\geq 1} \in \mathcal{T}^{\geq 1}$.

Note that if $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ is a t-structure on $\mathcal{T}$, then $\mathcal{T}^{\leq 0}$ is called the *aisle* and $\mathcal{T}^{\geq 0}$ is called the *coaisle*, and $\mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ is called the *heart* of this t-structure. We know that the inclusion functor $\mathcal{T}^{\leq 0} \to \mathcal{T}$ has a right adjoint and the inclusion functor $\mathcal{T}^{\geq 0} \to \mathcal{T}$ has a left adjoint and the heart is always abelian.

For my further research I plan to pursue the following directions:

A.1 Enhancements of triangulated categories

Despite the usefulness of triangulated categories, even from the beginning one can observe that there are some reasons for which the definition of a triangulated category is unsatisfactory. More precisely axioms defining a triangulated category are not functorial. Further, performing usual constructions with triangulated categories, that is taking various categories of functors landing in a triangulated category, the result is not more triangulated. A way of overcoming these inconvenient facts is to consider so called enhancements of triangulated categories. I plan to include in my study various enhancements of triangulated categories, e.g. differential graded (for short DG) categories, Quillen model categories, filtered enhancements, derivators in the sense of Grothendieck etc. Some of them I already considered as one can see from preprints [12] and [55].

Problem A.1.1. *Clarify the relationships between filtered enhancements and derivators.*

In the preprint [55] it is initiated this study by showing the following:

Theorem A.1.2. *Every triangulated category which is the underlying category of a stable derivator has a filtered enhancement.*

This result gives an affirmative answer to a conjecture due to Beĭlinson. The main reason for which filtered enhancement were defined seems to be they allow the construction of so called realization functor. More precisely if the triangulated category $\mathcal{T}$ has a filtered enhancement and we consider an arbitrary t-structure τ in $\mathcal{T}$ whose heart is the (abelian) category $\mathcal{A}$, then we can construct a functor $\texttt{real} : \mathbf{D}^b \to \mathcal{T}$ such that $\texttt{real} \circ \mathrm{H}^i_\tau = \mathrm{H}^i$, where H^i_τ are the cohomology functors associated to τ and H^i are the canonical cohomology functors in $\mathbf{D}^b$. Often the functor `real` is fully faithful. We intend to study how is this functor to be defined at the level of the derivators. The paradigm for these results should be the following:

Example A.1.3. If $\mathcal{A}$ is an abelian category and $X \mapsto \mathbf{D}(\mathcal{A}^X)$ is the derivator which associate to any small category X the derived category of the abelian category $\mathcal{A}^X$, then considering the category of (finite) filtered complexes over $\mathcal{A}^X$ we construct as usual (that is by formally inverting filtered quasi-isomorphisms) the filtered derived category $\mathbf{DF}(\mathcal{A}^X)$. We can assembly this in order to construct a new derivator $X \mapsto \mathbf{DF}(\mathcal{A}^X)$.

Problem A.1.4. *Formulate and prove a version of Brown representability at the level of various enhancements.*

An approach of this problem at the level of Quillen model categories can be found in Hovey's paper [33]. There it is proved a homological version of Eilenberg–Watts theorem that characterize tensor product. It would be interesting to find an analogous results at the more general level of derivators.

A.2 Generalizations of tilting theory

It is well–known that if T is a small tilting R-module (where R is a ring), then T induces equivalence of categories between $\mathbf{D}(\mathrm{Mod}(R))$ and $\mathbf{D}(\mathrm{Mod}(E))$ where $E = \mathrm{End}_R(T)$ is the endomorphism ring of T. If we drop the smallness assumption, then we still can manage the situation in order to obtain an equivalence between $\mathbf{D}(\mathrm{Mod}(R))$ and a localization of $\mathbf{D}(\mathrm{Mod}(E))$. This fact gives the possibility to define tilting objects and various generalizations of them at the level of an abstract triangulated category. We plan to study some of these generalizations as for example silting objects. A *silting object* in a triangulated category

T was defined in [69], as being an object $T \in \mathcal{T}$ such that $(T^{\perp_{>0}}, T^{\perp_{<0}})$ is a t-structure in $\mathcal{T}$ and $T \in T^{\perp_{>0}}$. Here we denote, for any subset $I \subseteq \mathbb{Z}$:

$$T^{\perp_I} = \{X \in \mathcal{T} \mid \mathcal{T}(T, \Sigma^i X) = 0 \text{ for all } i \in I\}.$$

Dually it is defined a *cosilting object*, by requiring that $({}^{\perp_{>0}}T, {}^{\perp_{<0}}T)$ is a t-structure in $\mathcal{T}$ and $T \in {}^{\perp_{>0}}T$.

Problem A.2.1. *In the case of silting formulate and prove a theorem analogous to the celebrated Tilting Theorem.*

Recall that if $T \in \mathrm{Mod}(R)$ is a small tilting module of projective dimension at most 1 and $E = \mathrm{End}_R(T)$, then the Tilting Theorem of Brenner and Butler (see [13]) says that there are torsion classes $(\mathcal{U}, \mathcal{V})$ in $\mathrm{Mod}(R)$ and $(\mathcal{X}, \mathcal{Y})$ in $\mathrm{Mod}(E)$ and mutually inverse equivalences:

$$\mathrm{Hom}_R(T, -) : \mathcal{U} \leftrightarrows \mathcal{Y} : - \otimes_E T \text{ and } \mathrm{Ext}_R(T, -) : \mathcal{V} \leftrightarrows \mathcal{X} : \mathrm{Tor}^E(-, T).$$

The theorem was generalized for tilting objects with arbitrary finite projective dimension (see [58]). Various other generalizations are also available. I want to see in which form this theory generalizes for the case of a silting object (complex). A particular case, namely for a silting complex of length 2, is studied in the preprint [11].

Problem A.2.2. *Clarify the relationship between the existence of a small (or big) silting object T in a triangulated category $\mathcal{T}$ and the existence of an equivalence between $\mathcal{T}$ and (a localization of) $\mathbf{D}(\mathrm{Mod}(E))$, where E is the DG-endomorphism algebra of T.*

Note that a particular case of this Problem, namely for $\mathcal{T} = \mathbf{D}(\mathrm{Mod}(R))$, where R is a ring, is studied in [12], but the same results were already obtained in [66].

Problem A.2.3. *Consider a "nice" cosilting object $T \in \mathcal{T}$. Does T still induce an equivalence or a duality between (a subcategory of) $\mathcal{T}$ and a localization of $\mathbf{D}(\mathcal{G})$, for a suitable DG-category $\mathcal{G}$?*

Observe that starting with a cotilting R-module in [75] it is constructed an abelian category $\mathcal{G}$ and an equivalence between $\mathbf{D}(\mathrm{Mod}(R))$ and $\mathbf{D}(\mathcal{G})$. It is interesting to see what is happen if the cotiling module is replaced by a cosilting complex or module. This problem is a a little more speculative, because I have less examples. However, I just speculated that it is possible, as in the case of passing from tilting to silting, to get analogous results, when we replace the usual category with one enriched over complexes, that is a DG-category.

A.3 Approximations and adjoints

As we have seen Brown representability can be used for constructing adjoints, but it is not the unique way. Another possibility is to use [63, Proposition 1.4] (see Corollary 2.2.5 for a particular case): If $\mathcal{T}$ is a triangulated category, then the inclusion functor $\mathcal{T}' \to \mathcal{T}$ of from a triangulated subcategory $\mathcal{T}'$ to $\mathcal{T}$ has a right (left) adjoint if and only if $\mathcal{T}'$ is precovering (preenveloping). In this point the approximation theory in abelian categories, that is the theory of cotorsion pairs, developed for example in [27], helps us to construct precovers and/or preenvelops.

Problem A.3.1. *Let $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair in a "sufficiently nice" abelian or exact category $\mathcal{A}$. Construct (co)resolutions of complexes over $\mathcal{A}$ with respect to this cotorsion pair. As in the case of well-known homotopically projective (injective) (co)resolutions for complexes of modules, derive the existence of some adjoints for the inclusion functors $\mathbf{K}_{\mathcal{X}-ac}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$, respectively $\mathbf{K}^{\mathcal{Y}-ac}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$, where $\mathbf{K}_{\mathcal{X}-ac}(\mathcal{A})$ and $\mathbf{K}^{\mathcal{Y}-ac}(\mathcal{A})$ are the homotopy categories of those complexes which become exact after application of the functor $\mathcal{A}(X, -)$, respectively $\mathcal{A}(-, Y)$.*

This Problem generalizes the approach in [18] and a positive solution would lead to a generalization of the equivalence between $\mathbf{K}(\mathrm{Proj}(R))$ and $\mathbf{K}(\mathrm{Inj}(R))$ for a Notherian ring R. This equivalence was first proved by Iyengar and Krause in [34] and it is an extension of the Grothendieck duality. Our approach should also be compared with Hovey's method of producing model categories structures out of cotorsion pairs (see [32]).

Problem A.3.2. *Develop an ideal approximations theory in triangulated categories.*

An ideal approximation theory is a theory where the usual cotorsion pair is replaced by a pair of ideals which are orthogonal to each other with respect to Ext-functor. Note that at the level of module (or even abelian) categories, such a theory is already done in [23] and [24]. An abstract approach of this theory in the triangulated case is contained in the preprint [10]. Note that various important notions related to the triangulated structure, the most notable being perhaps the one of Toda bracket, appears naturally in the development of this theory. To explore these connections is a subject of a further research.

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